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On the enumeration of irreducible k-fold Euler sums and their roles in knot theory and field theory

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Abstract A generating function is given for the number, $E(l,k)$, of irreducible k-fold Euler sums, with all possible alternations of sign, and exponents summing to l. Its form is remarkably simple: $\sum_n E(k+2n,k)\,x^n = \sum_{d|k}\mu(d)\,(1-x^d)^{-k/d}/k$, where μ is the Möbius function. Equivalently, the size of the search space in which k-fold Euler sums of level l are reducible to rational linear combinations of irreducible basis terms is $S(l,k)=\sum_{n<k}\binom{\lfloor(l+n-1)/2\rfloor}{n}$. Analytical methods, using Tony Hearn's REDUCE, achieve this reduction for the 3698 convergent double Euler sums with $l\leq 44$; numerical methods, using David Bailey's MPPSLQ, achieve it for the 1457 convergent k-fold sums with $l\leq 7$; combined methods yield bases for all remaining search spaces with $S(l,k)\leq 34$. These findings confirm expectations based on Dirk Kreimer's connection of knot theory with quantum field theory. The occurrence in perturbative quantum electrodynamics of all 12 irreducible Euler sums with $l\leq 7$ is demonstrated. It is suggested that no further transcendental occurs in the four-loop contributions to the electron's magnetic moment. Irreducible Euler sums are found to occur in explicit analytical results, for counterterms with up to 13 loops, yielding transcendental knot-numbers, up to 23 crossings.

1 Introduction

Recent progress in number theory [1, 2] interacts strongly with the connection between knot theory and quantum field theory, discovered by Dirk Kreimer [3, 4], and intensively investigated to 7 loops [5] by analytical and numerical techniques. The sequence of irreducible non-alternating double Euler sums studied in [1] starts with a level-8 sum that occurs in the 6-loop renormalization of quantum field theory [5], where its appearance is related [4] to the uniquely positive 8-crossing knot 8_{19}; the sequence of irreducible non-alternating triple Euler sums in [2] starts with a level-11 sum that occurs at 7 loops, where its appearance is associated with the uniquely positive hyperbolic 11-crossing knot [5].

These exciting connections, between number theory, knot theory, and quantum field theory, led to work with Bob Delbourgo and Dirk Kreimer [6], on patterns [7] of transcendentals in perturbative expansions of field theories with local gauge invariance, and with John Gracey and Dirk Kreimer [8], on transcendentals generated by all-order [9] results in field theory, obtainable in the limit of a large number, N, of interacting fields [10, 11].

In the course of the large-N analysis [8], the number theory in [1] appeared to constitute a severe obstacle to the development of the connection between knot theory and field theory. From the skeining of link diagrams that encode the flow of momenta in Feynman diagrams, we repeatedly obtained a family of knots, associated with the occurrence of irreducible Euler sums in counterterms. The obstacle was created by the (indubitably correct) 'rule of 3' discovered in [1], for non-alternating sums $\zeta(a,b)=\sum_{n>m}1/n^a m^b$ of level $l=a+b$. The analysis of [1] shows that non-alternating sums of odd levels are reducible, while at even level $l=2p+2$ there are $\lfloor p/3\rfloor$ irreducibles, where $\lfloor\ldots\rfloor$ is the integer part. At levels 8 and 10 this made us very happy, since we had the 8-crossing knot 8_{19} to associate with $\zeta(5,3)$, and the 10-crossing knot 10_{124} to associate with $\zeta(7,3)$. Thereafter the knots increase in number by a 'rule of two', giving $\lfloor p/2\rfloor$ knots with $l=2p+4$ crossings. So there are two 12-crossing knots, while [1] has only *one* level-12 irreducible, and two 14-crossing knots, which *is* [1] the number of level-14 irreducibles.

Faced with a 12-crossing knot in search of a number, we saw two ways to turn: to study 4-fold non-alternating sums, or 2-fold sums with alternating signs. The first route is numerically intensive: it soon emerges that well over 100 significant figures are needed to find integer relations between 4-fold sums at level 12. The second route is analytically challenging; it soon emerges that at all even levels $l\geq 6$ there are relations between alternating double sums that cannot be derived from any of the identities given in [1].

Remarkably, these two routes lead, eventually, to the *same* answer. The extra 12-crossing knot is indeed associated with the existence of a 4-fold non-alternating sum, $\zeta(4,4,2,2)=\sum_{n>m>p>q}1/n^4m^4p^2q^2$, which *cannot* be reduced to non-alternating sums of lower levels. It is, equivalently, associated with the existence of an irreducible *alternating* double sum, $U_{9,3}=\sum_{n>m}\{(-1)^n/n^9\}\{(-1)^m/m^3\}$. The equivalence stems from the unsuspected circumstance that the combination $\zeta(4,4,2,2)-(8/3)^3\,U_{9,3}$, and only this combination, is reducible to non-alternating double sums. Moreover, $l=12$ is the lowest level at which the reduction of non-alternating 4-fold sums necessarily entails an alternating double sum. The 'problem pair' of knots are a problem no more. Their entries in the knot-to-number dictionary [12] record that they led to a new discovery in number theory: *the reduction of non-alternating sums necessarily entails alternating sums.*

This discovery led me to study the *whole* universe of k-fold Euler sums, with all possible alternations of sign, at all levels l. As will be shown in this paper, it is governed by beautifully simple rules, which might have remained hidden, were it not for Dirk Kreimer's persistent transformation of the Feynman diagrams of field theory to produce a *pair* of 12-crossing knots.

The remainder of the paper is organized as follows. Section 2 states[1] the formula for the number, $E(l,k)$, of irreducible k-fold sums, with all possible alternations of sign, at level l. Section 3 outlines the process by which it was discovered. The anterior numerics of Section 4 describe the high-precision evaluation methods and integer-relation searches that helped to produce the formula; the posterior analytics of Section 5 describe computer-algebra proofs of rigorous upper bounds on $E(l,k)$ that are respected (and often saturated) by it. Section 6 summarizes numerical and analytical findings by listing an instructive choice of concrete bases. Section 7 shows that all 12 of the irreducible sums with $l \le 7$ appear in perturbative quantum electrodynamics. Section 8 considers the import, for quantum field theory, for knot theory, and for number theory, of results obtained by calculations up to level 23, corresponding to knots with up to 23 crossings, and to Feynman diagrams with up to 13 loops.

2 Result

To specify an alternating k-fold Euler sum, one may give a string of k signs and a string of k positive integers. It is very[2] convenient to combine these strings, by defining

$$\zeta(a_1,\ldots,a_k)=\sum_{n_i>n_{i+1}}\prod_{i=1}^{k}\frac{(\operatorname{sign}a_i)^{n_i}}{n_i^{|a_i|}}\,, \tag{1}$$

for $k>1$, on the *strict* understanding that the arguments are non-zero integers, and that $a_1\neq 1$, to prevent a divergence. Hence one avoids a proliferation of disparate symbols for the 2^k types of k-fold sum. The correspondence with the double-sum notations of [1] is

$$\begin{aligned}
\zeta(s,t) &= \sigma_{\rm h}(t,s)\,, && (2)\\
\zeta(-s,t) &= \alpha_{\rm h}(t,s)\,, && (3)\\
\zeta(s,-t) &= -\sigma_{\rm a}(t,s)\,, && (4)\\
U_{s,t}\equiv\zeta(-s,-t) &= -\alpha_{\rm a}(t,s)\,, && (5)
\end{aligned}$$

for *positive* integers s and t, with emphatically *no* implication of analytic continuation.

In perturbative quantum field theory, three-loop radiative corrections [13, 14] involve

$$U_{3,1}=\sum_{n>m>0}\frac{(-1)^{n+m}}{n^3m}=\tfrac12\zeta(4)-2\left\{\mathrm{Li}_4\left(\tfrac12\right)+\tfrac{1}{24}\ln^2 2\left(\ln^2 2-\pi^2\right)\right\}\,. \tag{6}$$

At six [15, 16, 17] and seven [5] loops, counterterms associated with the $(4,3)$ and $(5,3)$ torus knots, 8_{19} [4] and 10_{124} [5], involve $U_{5,3}$ and $U_{7,3}$, whose irreducibility is equivalent to that of $\zeta(5,3)$ and $\zeta(7,3)$, respectively. In higher counterterms, the *independent* irreducibility of $U_{9,3}$ and $U_{7,5}$ is associated [8] with a *pair* of 12-crossing knots.

[1] It would be very difficult to *prove*. One cannot even prove that $\zeta^2(3)/\zeta(6)$ is irrational.

[2] Without such a convention, results such as (22,30) become almost unreadable.

In what follows, the number of summations, k, is referred to as the *depth* of the sum (1), and $l = \sum_i |\,a_i|$ is referred to as its *level*. The set $S_{l,k}$ of convergent sums of level l and depth k has

$$N(l,k) = 2^k \binom{l-1}{k-1} - 2^{k-1} \binom{l-2}{k-2} \tag{7}$$

elements, as is easily proven by induction. The number of convergent sums at level l is

$$N(l) = \sum_{k=1}^{l} N(l,k) = 4 \times 3^{l-2}\,, \tag{8}$$

provided that $l > 1$.

Table 1: Euler's triangle[3] of irreducibles, at level l and depth k, for $l + k \leq 32$.

$l\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
1	1															
2	1															
3	1															
4		1														
5	1		1													
6		1		1												
7	1		2		1											
8		2		2		1										
9	1		3		3		1									
10		2		5		3		1								
11	1		5		7		4		1							
12		3		8		9		4		1						
13	1		7		14		12		5		1					
14		3		14		20		15		5		1				
15	1		9		25		30		18		6		1			
16		4		20		42		40		22		6		1		
17	1		12		42		66		55		26		7		1	
18		4		30		75		99		70		30		7		...
19	1		15		66		132		143		91		35		...	
20		5		40		132		212		200		112		...		
21	1		18		99		245		333		273		...			
22		5		55		212		429		497		...				
23	1		22		143		429		715		...					
24		6		70		333		800		...						
25	1		26		200		715		...							
26		6		91		497		...								
27	1		30		273		...									
28		7		112		...										
29	1		35		...											
30		7		...												
31	1		...													

[3]It seems appropriate to call it *Euler's* triangle; see Section 8.3 for the connection with *Pascal's*.

The main purpose of this work is to determine, as in Table 1, the number, $E(l,k)$, of irreducible sums in $S_{l,k}$, i.e. the minimum number of sums which, together with sums of lesser depth and products of sums of lower level, furnish a basis for expressing the elements of $S_{l,k}$ as linear combinations of terms, with *rational* coefficients. The conclusion is that

$$E(l,k) = \delta_{l,2}\delta_{k,1} + \frac{2}{l+k}\sum_{2d|l\pm k}\mu(d)\binom{\frac{l+k}{2d}}{\frac{l-k}{2d}}\,, \tag{9}$$

where the summation is over the positive integers d such that $(l \pm k)/2d$ are integers, and is weighted by the Möbius function, $\mu(d)$, which vanishes if d is divisible by the square of a prime and otherwise is ± 1, according as whether d has an even or odd number of prime divisors. With the exception of $\ln 2$, from $S_{1,1}$, and π^2, from $S_{2,1}$, the irreducibles come from $S_{k+2j,k}$, with $j > 0$. Moreover, $S_{k+2j,k}$ contains the same number of irreducibles as $S_{j+2k,j}$, as illustrated in Table 1.

From (9) one obtains the number of irreducibles at level l:

$$E(l) = \sum_{k=1}^{l} E(l,k) = \frac{1}{l}\sum_{d|l}\mu(l/d)\left\{F_{d+1} + F_{d-1}\right\}\,, \tag{10}$$

in terms of the Fibonacci numbers. The integer sequence (10) is tabulated as M0317 by Sloane and Plouffe [18], who record its origin in the study [19] of congruence identities. It shows that the irreducibles are an exponentially decreasing fraction of the number of convergent sums at level l:

$$\frac{E(l)}{N(l)} \sim \frac{9}{4l}\exp(-\beta l)\,; \quad \beta = \ln\frac{6}{\sqrt{5}+1} \approx 0.6174\,. \tag{11}$$

This relative sparsity is illustrated in Table 2.

Table 2: The numbers, $E(l)$ and $N(l)$, of irreducibles and sums, at level l, for $l \leq 13$.

l	1	2	3	4	5	6	7	8	9	10	11	12	13
$E(l)$	1	1	1	1	2	2	4	5	8	11	18	25	40
$N(l)$	1	4	12	36	108	324	972	2916	8748	26244	78732	236196	708588

From (9) one may obtain the size, $S(l,k)$, of the search space for sums of level l and depth k, i.e. the minimum number of terms that allow one to express every element of $S_{l,k}$ as a linear combination, with rational coefficients. This basis consists of the irreducibles with level l and depths no greater than k, together with all the independent terms that are formed from products of sums whose levels sum to l and whose depths sum to no more than k.

One may generate $S(l,k)$ from $E(l,k)$, by expanding

$$\sum_{n=1}^{l_{\max}}\left[\sum_{l=1}^{l_{\max}}\left((X_{2,1,1}\,x^2)^l y + \sum_{k=1}^{l}\sum_{i=1}^{E(l,k)} X_{l,k,i}\,x^l y^k\right)\right]^n \tag{12}$$

to order $l_{\max}$ in both x and y, where $l_{\max}$ is greatest level required, and $X_{l,k,i}$ serves as a symbol for the ith irreducible in $S_{l,k}$, so that $(X_{2,1,1})^n$ stands for any rational multiple

of π^{2n}, and hence for $\zeta(2n)$. Then $S(l,k)$ is obtained by selecting the terms of order x^l, dropping powers of y higher than y^k, setting $y=1$, and counting the length of the resulting expression. This procedure is easily implemented in REDUCE, Maple, Mathematica, etc.

Table 3: The size, $S(l,k)$, of the search space for sums in $S_{l,k}$, for $k \le l \le 15$.

$l\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1														
2	1	2													
3	1	2	3												
4	1	3	4	5											
5	1	3	6	7	8										
6	1	4	7	11	12	13									
7	1	4	10	14	19	20	21								
8	1	5	11	21	26	32	33	34							
9	1	5	15	25	40	46	53	54	55						
10	1	6	16	36	51	72	79	87	88	89					
11	1	6	21	41	76	97	125	133	142	143	144				
12	1	7	22	57	92	148	176	212	221	231	232	233			
13	1	7	28	63	133	189	273	309	354	364	375	376	377		
14	1	8	29	85	155	281	365	485	530	585	596	608	609	610	
15	1	8	36	92	218	344	554	674	839	894	960	972	985	986	987

Inspection of Table 3 reveals that the sizes satisfy the recurrence relation

$$S(l+1,k) = S(l,k-1) + S(l-1,k)\,, \quad \text{for } l > k > 1\,. \tag{13}$$

Moreover, the Fibonacci numbers appear at maximum depth:

$$S(l,l) = S(l,l-1) + 1 = F_{l+1}\,. \tag{14}$$

From the Lucas relation between Fibonacci and binomial numbers, one obtains

$$S(l,k) = \sum_{n=0}^{k-1} \binom{\left\lfloor \frac{l+n-1}{2} \right\rfloor}{n}\,, \tag{15}$$

as the solution to (13,14).

Since the process of generating Table 3 from Table 1 is reversible, formula (9) may be replaced by the simple statement that at level l the size of the search space increases by the binomial coefficient

$$S(l,k+1) - S(l,k) = \binom{\left\lfloor \frac{l+k-1}{2} \right\rfloor}{k}\,, \tag{16}$$

when the depth is increased from k to $k+1$. This pragmatic formulation[4] is rather helpful, when using an integer-relation search algorithm, such as PSLQ [20]. An even simpler, though informal, restatement appears in Section 8.3.

[4] Computation leaves no doubt as to the equivalence of (9) and (16), though it is not yet proven.

3 Discovery

Some brief historical remarks seem in order at this stage, since my route to (9) in fact *began* with the observation of 7 members of the Fibonacci sequence (14), during the course of 800-significant-figure integer-relation searches entailed by the relation between knot theory and field theory.

Using David Bailey's magnificent MPPSLQ [21] implementation of PSLQ [20], I succeeded in reducing all 1457 sums in $\{S_{l,k} \mid 1 \le k \le l \le 7\}$ to just 12 numbers and their products. These irreducible numbers may conveniently be taken as

$$\ln 2,\ \pi^2,\ \{\,\zeta(l) \mid l = 3,5,7\},\ \{\,\alpha(l) \mid l = 4,5,6,7\},\ U_{5,1},\ \zeta(5,1,-1),\ \zeta(3,3,-1), \tag{17}$$

with the polylogarithms

$$\alpha(l) = \frac{(-\ln 2)^l}{l\,!}\left\{1 - \frac{l(l-1)}{12}\left(\frac{\pi}{\ln 2}\right)^2\right\} + \sum_{n=1}^{\infty}\frac{1}{2^n n^l} \tag{18}$$

populating the deepest diagonal of Table 1. As in (6), the definition (18) postpones the appearance of $\pi^2(\ln 2)^{l-2}$ and $(\ln 2)^l$ to depths $l-1$ and l, respectively, which is required by (16). Moreover $\alpha(1) = \alpha(2) = 0$, and $\alpha(3) = \frac{7}{8}\zeta(3)$, which is not a new irreducible.

After noting the Fibonacci sequence for the maximum sizes $S(l,l)$, with $l \le 7$, I sought a combinatoric form for $S(l,k)$. Formula (15) suggested itself on the empirical basis of the 28 cases with $1 \le k \le l \le 7$, and was then submitted to intense numerical and analytical tests at higher levels, as indicated in Sections 4 and 5. I implemented the generator (12), using the weight and length commands of REDUCE [22], and constructed Table 1, working backwards from (16). Next came the observation that the generating functions for the $k = 2$ and $k = 3$ columns of Table 1 have comparable forms:

$$G_2(x) = \tfrac{1}{2}\left\{1/(1-x)^2 - 1/(1-x^2)\right\}\,, \quad G_3(x) = \tfrac{1}{3}\left\{1/(1-x)^3 - 1/(1-x^3)\right\}\,. \tag{19}$$

The symmetry of Table 1 was vital to the discovery of the simple formula

$$G_k(x) = \sum_{n=0}^{\infty} E(k+2n,k)\,x^n = \frac{1}{k}\sum_{d|k}\mu(d)\,(1-x^d)^{-k/d}\,, \tag{20}$$

which produces formula (9). Computation of $\sum_k E(l,k)$, for $3 \le l \le 100$, revealed that it produced the integers nearest to $\sum_{d|l}\mu(l/d)\phi^d/l$, where $\phi = \frac{1}{2}(\sqrt{5}+1)$ is the golden section. The equivalent form (10) was obtained by submitting the integer sequence $E(3),\ldots,E(23)$ to Neil Sloane's helpful on-line [23] version of [18]. It was noteworthy that this lookup returned the values $E(1) = E(2) = 1$, from the Fibonacci form (10), agreeing with the appearance of $\ln 2$, at $l = 1$, and π^2, at $l = 2$. Their inclusion in the $k = 1$ column of Table 1, above Euler's triangle, thus became appropriate, as they appear to usher in the higher transcendentals, in much the same way that $F_1 = F_2 = 1$ seed the exponential growth of MPPSLQ's CPUtime (or Fibonacci's rabbits) along the deepest diagonal of Table 3.

It remains to describe yet more probing numerical and analytical tests, in further support of the claim in (16), and its equivalent version in (20). Nonetheless, it is hoped that the reader already shares some of my feeling that these two formulæ are simply too beautiful to be wrong.

4 Anterior numerics

4.1 Numerical evaluation

Suppose that one wishes to obtain a high-precision approximation to $S_\infty = \lim_{N\to\infty} S(N)$, where the truncated sum $S(N) = \sum_{n\le N} R(n)$ has a summand $R(n)$ with an asymptotic series in $1/n$, starting at $1/n^{C+1}$, with $C > 0$. From $\{S(n) \mid N - M \le n \le N\}$, one may form a table $\{T(n,m) \mid N - M + m \le n \le N,\ 0 \le m \le M\}$, by the procedure

$$T(n,m+1) = \frac{(C+m+n)\,T(n,m) - n\,T(n-1,m)}{C+m}\,, \tag{21}$$

with $T(n,0) = S(n)$. The method exploits the vanishing of $C+m+n-n/(1-1/n)^{C+m}$ as $n \to \infty$. It takes $M(M+1)/2$ applications of (21) to produce $T(N,M)$. Provided that $N >> M \ge 1$, and that rounding errors have been controlled, one obtains the approximation $S_\infty = T(N,M) + \mathrm{O}(M!/N^{M+C})$, where the factorial becomes significant for $N >> M >> C \ge 1$.

This elementary and economical method of accelerated convergence is applicable to every Euler sum of the form (1) that has no argument equal to unity, in which case repeated appeal to the Euler-Maclaurin formula [24] underwrites the absence of logarithms in the expansion of the truncation error, and the outermost summation is of the form $\sum_n \{R(2n) \pm R(2n+1)\}$, with $R(n) = \mathrm{O}(1/n^a)$, for an argument $a_1 = \pm a$. Thus one should store truncations after even increments of n_1, and set $C = a_1 - 1$ for $a_1 \ge 2$, or $C = |\,a_1|$ for $a_1 \le -1$, in procedure (21). To obtain the starting values, one has merely to set up a *single* loop that updates and stores each layer of the nest as its particular summation variable, n_i, assumes the even and odd values within the loop. Thus the evaluation time for a truncation at N of a k-fold sum is roughly proportional to kN.

For sums with no unit arguments, one needs therefore only a few lines of conventional FORTRAN, which may be handed over to David Bailey's TRANSMP [25] utility, to produce code that calls his MPFUN [26] multiple-precision subroutines. As a rule of thumb, the working accuracy should be somewhat better than the *square* of the desired output accuracy, when using (21). When, and *only* when, rounding errors are so controlled, an output accuracy of *very* roughly $M!/N^M$ is achieved by M iterations of (21), with input data obtained from looping over N pairs of successive even and odd integers. For a 4-fold sum, the accumulation of data takes roughly $50N$ calls of MPFUN subroutines, and the acceleration of the convergence takes roughly $8M^2$ calls. Thus, to achieve P significant figures for a 4-fold sum, one should choose a value of M that keeps the time factor, $T \approx 50(10^P M!)^{1/M} + 8M^2$, close to its absolute minimum. I find that between 780 and 800 significant figures are reliably and efficiently achieved with $M = 440$ iterations, and truncation at $N = 10^4$, which entails $T \approx 2 \times 10^6$ calls to MPFUN subroutines, operating at multiple precision 1700. This takes less than half an hour on a DEC Alpha 3000-600 machine, corresponding to a call rate that is faster than 1 kHz. The memory requirement is less than 1 MB: an array of 440×240 4-byte cells holds the truncations in multiple precision and is updated iteratively by (21).

Of course, the above method *fails* as soon as one sets any of the arguments to unity. However, all is not lost. By iteratively applying the Euler-Maclaurin formula, one arrives

at the conclusion that the maximum power of $\ln n$ in the truncation error $\sum_{i,j} A_{i,j}(\ln n)^i/n^j$ is the *largest* number, d, of *successive* units in the string of arguments, since only for $j=1$ does the integration of $(\ln x)^i/x^j$ increase the power of $\ln x$. Consider, for example, $\zeta(a,1,b,1)$, with $a\neq 1$ and $b\neq 1$. By the time the logarithm generated by the n_4 summation is felt by the n_2 summation, it has acquired an inverse power of n_2, thanks to the benignity of the Euler-Maclaurin formula for the n_3 summation. Thus this sum has *demon*-number $d=1$, even though it contains two unit arguments. On the other hand, $\zeta(a,b,1,1)$ and $\zeta(a,1,1,b)$ have $d=2$, as does $\zeta(a,1,1,b,1,a,1,1)$, for example. At level l the most demonic convergent sum has unity for all its arguments, except for the first, which must therefore be $a_1=-1$. Fortunately, it is possible to give exact expressions for this beast, with $d=l-1$, and another, with $d=l-2$. With a string of n unit arguments denoted by $\{1\}_n$, the all-level results

$$\zeta(-1,\{1\}_{l-1})=\frac{(-\ln 2)^l}{l\,!}\,,\quad -\zeta(-1,-1,\{1\}_{l-2})=\mathrm{Li}_l\left(\tfrac{1}{2}\right)=\sum_{n=1}^{\infty}\frac{1}{2^n n^l}\,, \tag{22}$$

were inferred numerically, and then obtained analytically, by Jon Borwein and David Bradley, as the coefficients of t^l in $G(t,-1)$ and $t\int_0^1 \mathrm{d}z\,G(t,z)/(1+z)$, generated by the trivially summable hypergeometric series $G(t,z)=tz\,{}_2F_1(1+t,1;2;z)=(1-z)^{-t}-1$. From $t\int_0^1 \mathrm{d}z\,G(t,z)/z$ one generates the corollary, $\zeta(2,\{1\}_{l-2})=\zeta(l)$, of a theorem [27]

$$\sum_{a_i>\delta_{i,1},\ l=\Sigma_i a_i}\zeta(a_1,a_2,\ldots,a_k)=\zeta(l)\,, \tag{23}$$

from Andrew Granville, which was used to check evaluations of non-alternating sums.

To mitigate the computational difficulties caused by unit arguments, truncation errors may be obtained analytically for *combinations* of Euler sums of the form

$$S(a;b_1,\ldots,b_{k-1})=\sum_{n=1}^{\infty}\frac{(\mathrm{sign}\,a)^n}{n^{|\,a|}}\prod_{i=1}^{k-1}\sum_{m_i=1}^{n-1}\frac{(\mathrm{sign}\,b_i)^{m_i}}{m_i^{|\,b_i|}}\,. \tag{24}$$

It seems appropriate to call (24) a *boxed* sum, since the symmetrical inner summations span a lattice that is confined to a $(k-1)$-dimensional hypercube by the outer summation variable. It is built out of symmetrical combinations of Euler sums with depths no greater than k, and is, so to speak, a 'cheap boxed set', available at a reduced [28] computational price, since it requires only the *multiplication* of $k-1$ polygamma Euler-Maclaurin series, followed by a *single* further application of the Euler-Maclaurin formula, to determine the truncation error to any order for which ones favorite computer-algebra engine has the power to multiply, differentiate, and integrate double series in $\ln n$ and $1/n$. Only depth-1 data is needed, since the inner summations can be rewritten as $\sum_{m_i}-\sum_{m_i\geq n}$, with the first term giving a constant and the second an asymptotic series in $1/n$, except for $b_i=1$, which gives Euler's constant, a log, and an asymptotic series.

This Euler-Maclaurin method is the obvious generalization of that used in [28], in the restricted cases with $b_i=1$, or $b_i=-1$. In the general case, much computer-algebra time may be consumed by multiplying asymptotic series for the distinct values of b_i and then integrating a long expression involving many powers of $\ln n$ and $1/n$, though the algorithm is straightforward to implement. To obtain a few hundred significant figures in a short time one may stay within REDUCE, without feeding thousands of lines of FORTRAN

statements to TRANSMP; at higher precision one benefits from this transfer of function, at the expense of non-trivial file management, when handling many boxed sums.

From an analytical point of view, it is highly significant that every 3-fold Euler sum can be transformed into a boxed sum: $\zeta(a_1,a_2,a_3)\simeq -S(a_2;a_1,a_3)$, where $\simeq$ will henceforward stand for 'equality modulo terms of lesser depth, their products, and consequent questions of convergence'. The argument is simple [2]: to the summation with $N\ge n_1>n_2>n_3\ge 1$, for the truncation of $\zeta(a_1,a_2,a_3)$, add that with $N\ge n_2>n_{1,3}\ge 1$, for the truncation of $S(a_2;a_1,a_3)$. Adding the case $n_1=n_2$, which has lesser depth, one has covered the values $N\ge n_1\ge 1$, and hence has a product of truncated sums of lesser depth. Thus $\zeta(a_1,a_2,a_3)+S(a_2;a_1,a_3)\simeq 0$. The questions of convergence obviously concern the case with $a_2=1$, in the limit $N\to\infty$. Such issues are handled with great dexterity in [2], in the case of non-alternating triple sums. Dealing, as now, with truncated sums, no problem of convergence arises. The snag is that one must multiply truncations to obtain the product term, so the process becomes both messy and *ad hoc*, from the perspective of a programmer seeking a systematic algorithm for sums of any depth. From a numerical point of view, little is gained from boxability at depth $k=3$.

Truly nested Euler sums (1) have depth $k\ge 4$, where only combinations of them can be boxed. In the next subsection it will be shown that at $k=4$ there first occurs a significant phenomenon, of central relevance to the claim of Section 2, and to knot-theoretical studies [3, 4, 5, 8]. But first, there is an outstanding computational dilemma to confront. Should one attempt to automate the process of *chaining* applications of the Euler-Maclaurin formula, feeding in the required values for *multiple* Euler sums from runs of lesser depth, and carefully separating the odd and even summations, at each link of a chain whose length one would like to vary? Or should one, rather, use the empirical truncation data to accelerate the convergence? In the absence of logs, the one-line procedure (21) settles the issue, to my mind. Why use masses of computer algebra to feed into TRANSMP information that is already sitting in the numerical data, in such immediately usable form? When there are logs, from unit arguments, the matter is moot. Having taken the empirical approach when the going was easy, I opted to stick with it, when the going got tough, at depths $k\ge 4$, with $d>0$ demons.

It is clear that high-precision knowledge of $K(d,M)=1+(d+1)M$ truncated values, for a sum with demon-number d, should suffice to accelerate the convergence by a factor of roughly $M!/N^M$, as was achieved with (21), for $d=0$. How best to achieve this, for $d>0$, is another matter. Unable to devise an easily programmable iterative method like (21), I returned to the brute-force method of using $K(d,M)$ truncations to solve directly for $K(d,M)$ unknowns, as was done a decade ago [15] in the investigations which first suggested the appearance of irreducibles with depth $k>1$ at the six-loop level of renormalization of quantum field theory, a prediction amply confirmed by recent six- and seven-loop analysis [5], and illuminated by knot theory [3, 4].

After translating the Gauss-Jordan [29] method into MPFUN calls, one may systematically obtain the $K(d,M)$ coefficients, in the approximation

$$S(n)\approx S_\infty+\sum_{i=0}^{d}(\ln n)^i\sum_{j=C}^{C+M-1}\frac{A_{i,j}}{n^j}\,,\qquad(25)$$

from $K(d,M)$ truncations, and in particular find an accurate value for S_∞. I chose the

truncations $\{S(N+pL) \mid 0 \le p \le (d+1)M\}$, taken distances L apart, starting at some value N. The choices of M, N, L, and working accuracy, to achieve a desired output accuracy for a sum with demon-number d, are an art learnt by experience, not yet a science that is fit to be explained here. Suffice it to say that it is wise to work at the *cube* of the desired accuracy, and that solving 10^3 sets of 10^3 equations, at a working accuracy of one part in 10^{10^3}, in order to perform 10^3 Euler-sum searches with the MPPSLQ [21] integer-relation finder, is demanding of core memory, CPUtime, vigilance, and patience. The rewards, in increased *analytical* understanding, are considerable.

4.2 Exact numerical results

By an exact numerical result, I mean an equation whose exactness is beyond intelligent doubt, yet is validated, to date, only by *very* high precision numerical evaluation. It is, strictly speaking, *possible* that an equation presented here is a divine hoax, just as the rationality of $\zeta^2(3)/\zeta(6)$ is still a possibility. The improbability beggars all description.

The exact numerical result that sparked the genesis of Tables 1 and 3, and revealed their utter simplicity, was found at level $l = 12$ and depth $k = 4$. It reads

$$\begin{aligned}
2^5 \cdot 3^3\,\zeta(4,4,2,2) &= 2^5 \cdot 3^2\,\zeta^4(3) + 2^6 \cdot 3^3 \cdot 5 \cdot 13\,\zeta(9)\,\zeta(3) + 2^6 \cdot 3^3 \cdot 7 \cdot 13\,\zeta(7)\,\zeta(5) \\
&+ 2^7 \cdot 3^5\,\zeta(7)\,\zeta(3)\,\zeta(2) + 2^6 \cdot 3^5\,\zeta^2(5)\,\zeta(2) - 2^6 \cdot 3^3 \cdot 5 \cdot 7\,\zeta(5)\,\zeta(4)\,\zeta(3) \\
&- 2^8 \cdot 3^2\,\zeta(6)\,\zeta^2(3) - \frac{13177 \cdot 15991}{691}\,\zeta(12) \\
&+ 2^4 \cdot 3^3 \cdot 5 \cdot 7\,\zeta(6,2)\,\zeta(4) - 2^7 \cdot 3^3\,\zeta(8,2)\,\zeta(2) - 2^6 \cdot 3^2 \cdot 11^2\,\zeta(10,2) \\
&+ 2^{14}\,U_{9,3}
\end{aligned} \tag{26}$$

and the sting is in its tail.

Exceptionally, factorizations of rationals were written in (26), with a central dot (not to be confused with a decimal point) denoting multiplication. If one dislikes the number 691, one may remove it, using $\zeta(12) = 691\pi^{12}/(3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13)$. Factorizations were given above lest the reader found the unfactorized rationals implausible, on first encountering an exact numerical result obtained by MPPSLQ. The practice will not be continued.

It is apparent that one needs at least 100 significant figures to discover (26), because of the product of two 5-digit primes in the π^{12} term. MPPSLQ (almost always) finds the integer relation, $\sum_i n_i s_i \approx 0$, with smallest euclidean norm, $(\sum_i n_i^2)^{1/2}$, consistent with the requested accuracy of fit. If one knew $\zeta(4,4,2,2)$ to only 100 significant figures, the routine would be at perfect liberty to return 13 'random' 8-digit integers that happened to fit at 100 significant figures. It would not care that the true form has attractive factorizations of all integers save one. At 800 significant figures, which is the accuracy to which (26) has been validated, the probability of it being spurious is of order 10^{-700}.

The import of (26) is dramatic: non-alternating sums, with exclusively positive arguments in (1), do *not* inhabit a cosy little world of their own, uncontaminated by contact with their alternating cousins, as the presence of $U_{9,3}$ clearly demonstrates.

As explained in the introduction, this was wonderful news for the connection between knot theory and quantum field theory [3, 4, 5, 8]. It was also what sparked the present

systematic inquiry into the *whole* universe of Euler sums of the form (1), setting firmly aside the notion that there is something special about non-alternating sums. The liberating effect is apparent in the discovery of the simple formulæ (16,20), which give order to the larger universe so embraced.

Inspection of Table 3 reveals some practical limits to exploration. Even with 800 significant figures one might expect to encounter problems finding relations in $S_{15,3}$, or $S_{10,4}$, each of which has 36 basis terms. If just one of the 37 integers in a desired integer relation exceeds 10^{23}, then MPPSLQ may (quite properly) fit 800-significant-figure data with 37 'random' 22-digit integers. Experience shows that integers of order 10^{22} are produced by successful searches in $S_{13,3}$, with 28 basis terms. So investigation of $S_{15,3}$ was judged to be imprudent, with 'only' 800 significant figures at hand. Because of the importance of level 10 in [5], investigation of $S_{10,4}$ is reported in Section 6.

The successful fit at all levels up to and including $l = 7$, using the 12 numbers (17), has been reported, as have the all-level results (22). There remains the apparently trifling matter of an exact numerical relation in $S_{6,2}$:

$$U_{4,2} = \tfrac{97}{96}\,\zeta(6) - \tfrac{3}{4}\,\zeta^2(3)\,, \tag{27}$$

with coefficients that Euler could, no doubt, have found by mental arithmetic. Surprisingly, nothing in the most recent work on double sums [1] suggested the existence of such a relation. It turned out to be relatively easy to devise an analytical proof, when thus apprised of its need, so the subject is postponed to the next section, where the armory of analytical tools is augmented. Yet the relation belongs here, since it was David Bailey's engine that disclosed it. The analytical techniques, developed to derive (27), feed back useful information for further numerical analysis.

First, they clear up the whole of the double-sum sector, for good and all, confirming the knot-theoretic expectation of a 'rule of 2', with $E(2p,2) = \lfloor p/2 \rfloor$, and thus allowing numerical exploration to progress to $k \geq 3$. Secondly, they provide rigorous (though non-optimal) upper bounds:

$$E(7,3) \leq 3\,, \quad E(8,4) \leq 4\,, \quad E(9,3) \leq 6\,, \quad E(11,3) \leq 11\,, \quad E(13,3) \leq 17\,. \tag{28}$$

Thirdly, for each bound it is possible to find an overcomplete basis, whose size is determined by the bound. Thus one needs to evaluate only a small fraction of the sums in these sectors, and then use MPPSLQ to reduce an overcomplete basis to a minimal basis of size $S(l,k)$. If the formula for $S(l,k)$ were false, MPPSLQ might sometimes fail to reduce the basis down to the claimed size, or might reduce it down to a size smaller than that claimed by (15). Of course, it never did the latter, else the claim would not have been made. The fact that it never did the former, with 800-significant-figure data, in search spaces of sizes up to 36, is a testament to its author [21] as well as to the formula. Finally, and most fortunately, it is possible to construct overcomplete bases, in the spaces bounded by (28), that are demon-free. Hence 800 significant figures are available, in less than half an hour per sum, at the touch of button (21).

Thanks to these circumstances, all Euler sums in $\{S_{l,3} \mid l \leq 14\}$ and $\{S_{l,4} \mid l \leq 9\}$ have been shown to be reducible to bases of the sizes given in Table 3, by the operation of MPPSLQ on overcomplete bases. Relatively simple examples of such reductions, in $S_{9,3}$,

$S_{11,3}$ and $S_{8,4}$, are provided by:

$$\begin{aligned}
\zeta(3,-3,-3) &= 6\,\zeta(5,-1,-3)+6\,\zeta(3,-1,-5)-\tfrac{315}{32}\ln 2\,\zeta(3)\,\zeta(5)+6\,U_{5,1}\,\zeta(3)\\
&+ \tfrac{40005}{128}\,\zeta(2)\,\zeta(7)-\tfrac{39}{64}\,\zeta^3(3)+\tfrac{1993}{256}\,\zeta(3)\,\zeta(6)+\tfrac{8295}{128}\,\zeta(4)\,\zeta(5)-\tfrac{226369}{384}\,\zeta(9),\\
\zeta(3,-5,-3) &= \tfrac{1059}{80}\,\zeta(5,3,3)+15\,\zeta(7,-1,-3)+15\,\zeta(3,-1,-7)+\tfrac{701}{69}\,U_{5,3}\,\zeta(3)\\
&+ 15\,U_{7,1}\,\zeta(3)-\tfrac{6615}{256}\ln 2\,\zeta(3)\,\zeta(7)-\tfrac{11852967}{2560}\,\zeta(11)+\tfrac{301599}{128}\,\zeta(2)\,\zeta(9)\\
&- \tfrac{124943}{5888}\,\zeta^2(3)\,\zeta(5)+\tfrac{1753577}{35328}\,\zeta(3)\,\zeta(8)+\tfrac{2960103}{5120}\,\zeta(4)\,\zeta(7)+\tfrac{3405}{32}\,\zeta(5)\,\zeta(6),\\
\zeta(3,-1,3,-1) &= \tfrac{61}{27}\,\zeta(-3,-3,-1,-1)-\tfrac{14}{3}\,\zeta(-5,-1,-1,-1)-\tfrac{185}{27}\,U_{5,1}\,\zeta(2)\\
&- \tfrac{163499}{22356}\,U_{5,3}+\tfrac{2051}{54}\,U_{7,1}+\tfrac{28}{9}\ln^2 2\,U_{5,1}+\tfrac{35}{96}\ln^2 2\,\zeta^2(3)\\
&- \tfrac{581}{64}\ln^2 2\,\zeta(6)-\tfrac{8735}{576}\ln 2\,\zeta(2)\,\zeta(5)-\tfrac{903}{64}\ln 2\,\zeta(3)\,\zeta(4)\\
&- \tfrac{1441}{288}\,\zeta(2)\,\zeta^2(3)+\tfrac{10365875}{476928}\,\zeta(3)\,\zeta(5)+\tfrac{36916435}{1907712}\,\zeta(8).
\end{aligned}\tag{29}$$

In $S_{13,3}$, the relations are more complex, with the prime factor 102149068537421 appearing in one case. Nonetheless, the probability of a spurious fit is less than 10^{-200}, in all cases, and is often much less than this. The existence of further relations, forbidden by (20), cannot be disproved by numerical methods. The euclidean norms of such unwanted relations would, however, exceed those of the discovered relations by factors ranging between 10^{10} and 10^{20}, which makes it *extremely* implausible that the formula is in error in any of the spaces with $S(l,k) \leq S(8,8) = F_9 = 34$.

5 Posterior analytics

5.1 Analytical tools

The analysis of [1, 2] makes use of two very simple types of relation between Euler sums. In the general case of k-fold sums, with all possible alternations of sign, it is somewhat difficult to notate these relations, in all generality. To avoid cumbersome formulæ, terms that involve sums of lower depth, and their products, will be omitted, as in the case of $\zeta(a_1,a_2,a_3) \simeq -S(a_2;a_1,a_3)$.

The first type of relation involves permutations of arguments:

$$\begin{aligned}
0 &\simeq \zeta(a_1,a_2,a_3,\ldots,a_k)+\zeta(a_2,a_1,a_3,\ldots,a_k)+\zeta(a_2,a_3,a_1,\ldots,a_k)+\ldots\\
&+ \zeta(a_2,a_3,a_4,\ldots,a_1)\,.
\end{aligned}\tag{30}$$

The proof is trivial: by including all insertions of a_1 in the string $a_2,a_3,\ldots,a_k$, one obtains a combination of sums that differs from the product $\zeta(a_1)\,\zeta(a_2,a_3,\ldots,a_k)$ only by terms in which summation variables are equal, corresponding to sums of depth $k-1$. For 4-fold sums, such relations reduce a set of 24 possible permutations to a set of 9, when the arguments are distinct.

The second type of relation follows from use of the partial-fraction identity [1, 2]

$$\frac{1}{A^aB^b}=\sum_{s>0}\frac{1}{(A+B)^{a+b-s}}\left\{\binom{a+b-s-1}{a-s}\frac{1}{A^s}+\binom{a+b-s-1}{b-s}\frac{1}{B^s}\right\}\tag{31}$$

for positive integer a and b. To see how this is used, consider the product $\zeta(a)\zeta(b,c,d)$, with positive arguments. It may be written as $\sum_{n,p,q,r} 1/n^a(p+q+r)^b(q+r)^c r^d$, where each summation variable runs over all the positive integers. Now apply (31), setting $A=n$ and $B=p+q+r$. The second type of resulting partial fraction is of the form $\sum_{n,p,q,r} 1/(n+p+q+r)^{a+b-s}(p+q+r)^s(q+r)^c r^d$, which is an Euler sum. To the first type, apply (31) with $A=n$ and $B=q+r$. Its second terms are also Euler sums. To its first, apply (31) with $A=n$ and $B=r$. Each term so produced is an Euler sum. Thus one has obtained a relation for non-alternating sums. By including signs, 16 such relations can be generated. In general, one gets 2^k relations by $k-1$ applications of (31) for every set of k exponents that one chooses for the initial product of sums.

It can seen that there is no scarcity of trivially derivable relations between Euler sums. The notable achievement of [1] was to organize the relations between double sums in such a way as to prove the reducibility of all double sums of odd level. In [2] non-alternating triple sums of even level were proven to be reducible. It was conjectured that non-alternating sums of level l and depth k are reducible whenever $l+k$ is odd. The stronger claim made by (9) is that this applies to alternating sums as well. In the course of the present work, reducibility has been demonstrated, by a combination of analytical and numerical methods, for all odd values of $l+k$ such that $S(l,k) \le S(14,3) = 29$.

As remarked previously, the identities of [1] are insufficient to derive the simple relation (27). No tally was given in [1] of the numbers of alternating double sums left unreduced at even levels, though the tally $\lfloor p/3 \rfloor$ was made for non-alternating double sums at level $2p+2$. Using REDUCE, one easily discovers that the relations given in [1] allow reduction of double sums to the set $\{\, U_{n+2m,n} \mid \min(n,m) > 0\}$ and that no further reduction is possible without additional input. Using MPPSLQ, one easily discovers that a truly irreducible set is furnished by $\{\, U_{2a+3,2b+1} \mid a \ge b \ge 0\}$. Thus the relations derived in [1] are insufficient in a way that is very easy to state: they fail to relate the even cases $\{\, U_{2a+4,2b+2} \mid a \ge b \ge 0\}$ to the odd cases $\{\, U_{2a+3,2b+1} \mid a \ge b \ge 0\}$.

To remedy this failure, it was sufficient to derive the further[5] relation

$$\zeta(a,b)+\zeta(-a,-b) = \sum_{s>0}(a+b-s-1)!\left\{\frac{\zeta_A(a+b-s,s)}{(a-s)!\,(b-1)!}+\frac{\zeta_B(a+b-s,s)}{(b-s)!\,(a-1)!}\right\} \tag{32}$$

$$\zeta_A(a,b) = \zeta(a,b)+\zeta(-a,b)-2^{1-a}\left\{\zeta(a,b)+\zeta(a+b)\right\} \tag{33}$$

$$\zeta_B(a,b) = 2^{1-a}\zeta(a,b) \tag{34}$$

with $a>1$ and $b>1$. To prove (32), one writes $\zeta(a,b)+\zeta(-a,-b) = \sum_{m,n} 2/(2m+n)^a n^b$. Then (31), with $A=2m+n$ and $B=n$, yields (33,34), after some rearrangements.

5.2 Double sums in knot theory and field theory

Adjoining (32) to relations in [1], it was possible to use REDUCE to derive expressions for the 3698 double sums up to level 44, in terms of the 121 irreducible double sums

$$\{\, U_{2a+3,2b+1} \mid a \ge b \ge 0,\ a+b \le 20\}\,. \tag{35}$$

[5] Jon Borwein later told me that (32,33,34) were known to, though not used by, the authors of [1].

The family of positive knots [8] that gave rise to this investigation has braid[6] words

$$\{\sigma_2\sigma_1^{2a+1}\sigma_2\sigma_1^{2b+1}\mid a\geq b\geq 1\}\,, \tag{36}$$

whose enumeration satisfyingly matches that of the irreducibles (35).

Note that one omits the knots $\{\sigma_2\sigma_1^{2a+1}\sigma_2\sigma_1\mid a\geq 0\}$ from the tally of 3-braids in (36), since Reidermeister moves transform them to $\{\sigma_1^{2a+3}\mid a\geq 0\}$, which are the 2-braid torus knots, corresponding [4] to the depth-1 irreducibles $\{\zeta(2a+3)\mid a\geq 0\}$, whose occurrence has been studied to all [33, 34, 35, 36] orders in quantum field theory. Correspondingly, the Euler sums $\{\,U_{2a+3,1}\mid a\geq 0\}$ do not occur in counterterms[7] though they may appear in the finite parts of integrals obtained from Feynman diagrams, and hence in the relationships between physical quantities, such as the charge and magnetic moment of the electron.

In fact, two of the most impressive perturbative calculations [13, 14] of physical quantities in quantum field theory produce the polylogarithm $\mathrm{Li}_4\left(\frac12\right)$, with the precise combination of $(\ln 2)^4$ and $\pi^2(\ln 2)^2$ terms given in (6) for $U_{3,1}=\frac12\zeta(4)-2\,\alpha(4)$. The ρ-parameter [13] of electroweak theory entails, at three loops, the level-4 term of [38]

$$B_4=-\left\{8\,U_{3,1}+\tfrac52\zeta(4)\right\}+\mathrm{O}(\varepsilon) \tag{37}$$

in $4-2\varepsilon$ spacetime dimensions. The level-4 terms in the three-loop contributions to the anomalous magnetic moment of the electron, $\frac12(g-2)_{\mathrm{e}}$, are obtained from [14] as

$$-\left\{\tfrac{50}{3}\,U_{3,1}+\tfrac{13}{8}\zeta(4)\right\}(e/2\pi)^6\,, \tag{38}$$

where $-e$ is the electron's charge, in units of $(\varepsilon_0\hbar c)^{1/2}$. Thus the lowest-level double-sum irreducible, $U_{3,1}$, is prominent in quantum field theory, though absent from counterterms.

Having seen the importance of double Euler sums in quantum field theory, and their relation to knot theory, one should move on to 3-fold alternating sums, since these too occur in field theory, as will be shown in Section 7. Unfortunately, the tools are not yet available to derive analytically *all* the relations implied by (20), with $k>2$. One must, therefore, make do with analytical derivations of *most* of them.

5.3 Proven bounds

At depths 3 and 4, the rigorous bounds (28) were proven by implementing the permutation (30) and partial-fraction (31) procedures in REDUCE, using its solve command. The results are conveniently returned in terms of ARBCOMPLEX [22] variables equal in number to the undetermined sums. One may then use the output to form a proven, overcomplete, demon-free basis. This was achieved for all triple sums up to level 14, and all quadruple sums up to level 9, which is no small undertaking, as may be judged from the facts that two days of CPUtime were insufficient to solve for the 676 triple sums at level 15, and 128 megabytes of core memory were insufficient to solve for the 1120 quadruple sums at level 10. I recommend these sectors as test grounds for improved algorithms.

[6]For an introduction to knot theory, try [30], followed by [31, 32].

[7]For details of renormalization procedures, see [37], pending publication of [12].

In the case of triple sums up to level 14, overcomplete bases were constructed by adjoining the non-alternating [2] irreducibles $\{\zeta(5,3,3), \zeta(7,3,3), \zeta(5,5,3)\}$ to the set

$$\{\zeta(2n+3, -2m-1, -2p-1) \mid n+m+p \leq 4,\ \min(n,m,p) \geq 0\}\,, \tag{39}$$

which is also undetermined by the permutation and partial-fraction identities. Attempts to reduce this set analytically, by adding identities obtained in the manner of (32), were not successful. Therefore the most pressing challenge is to *prove* the MPPSLQ result that $E(7,3) \leq 2$, since only the bound $E(7,3) \leq 3$ has been obtained rigorously, from (39). In the case of quadruple sums up to level 9, the analytical method fails to find two MPPSLQ relations in $S_{8,4}$, one of which is given in (29). Hence I recommend the $S_{7,3}$ and $S_{8,4}$ sectors as places to start the hunt for more powerful analytical techniques.

6 Concrete bases

To aid the elucidation of (9,16) in Section 8.3, it is instructive to summarize the results of MPPSLQ, from Section 4, and REDUCE, from Section 5, by giving concrete irreducibles whose values and products span, *inter alia*, all those spaces of Table 3 with $S(l,k) < 36$. Also included is a choice of the 5 irreducibles in $S_{10,4}$, which proved attainable with MPPSLQ, despite the large size, $S(10,4) = 36$. To save space, only argument strings are given, with a bar denoting a negative argument, and hence an alternation of sign at the corresponding layer of the nest.

For $k = 1$, one obviously needs $\ln 2$, π^2, and the odd-zetas, with argument strings $\{(2n{+}1) \mid n > 0\}$. For $k = 2$, the argument strings $\{(\overline{2n+1}, \overline{2m+1}) \mid n > m \geq 0\}$ suffice, to level $l = 44$, and presumably for ever. For the remaining spaces with $S(l,k) < 36$, see Table 4, which also includes $S_{10,4}$.

Table 4: Concrete sets of argument strings, yielding demon-free minimal bases.

$S_{5,3}$	$(3,\bar{1},\bar{1})$						
$S_{7,3}$	$(5,\bar{1},\bar{1})$	$(3,\bar{1},\bar{3})$					
$S_{9,3}$	$(7,\bar{1},\bar{1})$	$(5,\bar{1},\bar{3})$	$(3,\bar{1},\bar{5})$				
$S_{11,3}$	$(9,\bar{1},\bar{1})$	$(7,\bar{1},\bar{3})$	$(5,\bar{1},\bar{5})$	$(5,\bar{3},\bar{3})$	$(3,\bar{1},\bar{7})$		
$S_{13,3}$	$(11,\bar{1},\bar{1})$	$(9,\bar{1},\bar{3})$	$(7,\bar{1},\bar{5})$	$(7,\bar{3},\bar{3})$	$(5,\bar{1},\bar{7})$	$(5,\bar{3},\bar{5})$	$(3,\bar{1},\bar{9})$
$S_{6,4}$	$(\bar{3},\bar{1},\bar{1},\bar{1})$						
$S_{8,4}$	$(\bar{5},\bar{1},\bar{1},\bar{1})$	$(\bar{3},\bar{3},\bar{1},\bar{1})$					
$S_{10,4}$	$(\bar{7},\bar{1},\bar{1},\bar{1})$	$(\bar{3},\bar{3},\bar{3},\bar{1})$	$(\bar{5},\bar{3},\bar{1},\bar{1})$	$(\bar{5},\bar{1},\bar{3},\bar{1})$	$(\bar{3},\bar{1},\bar{5},\bar{1})$		
$S_{7,5}$	$(3,\bar{1},\bar{1},\bar{1},\bar{1})$						
$S_{8,6}$	$(\bar{3},\bar{1},\bar{1},\bar{1},\bar{1},\bar{1})$						

It is emphasized that the choices of Table 4 are *not* the most efficient for integer-relation searches; other choices, as in (17), result in smaller euclidean norms from successful searches. However, one learns more from the patterns of Table 4 than from the economic choices of (17).

First, note that only odd arguments are required: the pattern begun by odd-zetas at depth 1 persists. Secondly, note that all arguments may taken as negative, save the first,

when the depth, k, is odd. Since the first argument cannot be unity, demons are thereby eliminated from the irreducibles. Each is thus computable, to 800 significant figures, in half an hour, using (21). Finally, note that not all permutations of odd integers occur.

Whether these features persist in larger search spaces, where MPPSLQ may well need more than 800 significant figures, can only be conjectured. The argument of Section 8.3, however, lends weight to the belief that the key to understanding irreducibility, at level l and depth k, lies in an analysis of a *restricted* set of permutations of partitions of l into precisely k positive *odd* integers.

7 Euler sums in quantum electrodynamics

To exemplify the utility and accuracy of the database of exact result for all sums with $l \leq 7$, it was used to evaluate the Laurent expansion of the three-loop on-shell charge-renormalization [38] constant of dimensionally regularized [39] quantum electrodynamics, up to terms of level 7, which may be taken as indicative of the transcendentality content of four-loop contributions to the anomalous magnetic moment of the electron.

Like the three-loop corrections [13] to the ρ-parameter of electroweak theory, on-shell charge renormalization involves [38]

$$\begin{aligned} B_4 &= \frac{(\mu-1)(\mu-\frac{3}{2})}{\pi^{3\mu}\Gamma^3(3-\mu)} \int\int\int \frac{\mathrm{d}^{2\mu}p\,\mathrm{d}^{2\mu}q\,\mathrm{d}^{2\mu}r}{(p^2+1)(q^2+1)(r^2+1)(p-q)^2(p-r)^2} \\ &\qquad \times \left\{ \frac{1}{(p-q-r)^2+1} - \frac{1}{(q-r)^2} \right\}, \end{aligned} \tag{40}$$

which is a difference of three-loop massive bubble diagrams, regularized in $2\mu \equiv 4 - 2\varepsilon$ euclidean spacetime dimensions. In [38] it was reduced to a ${}_3F_2$ hypergeometric series [40]:

$$\begin{aligned} B_4 &= \frac{7}{24\varepsilon^4}\left\{1 - \frac{\Gamma(1-\varepsilon)\,\Gamma^2(1+2\varepsilon)\,\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)\,\Gamma(1+4\varepsilon)}\right\} - \frac{\pi^2}{3\varepsilon^2}\,\frac{\Gamma(1+2\varepsilon)\,\Gamma(1+3\varepsilon)}{2^{6\varepsilon}\,\Gamma^5(1+\varepsilon)} \\ &\quad + \frac{8}{3\varepsilon^2(1+2\varepsilon)}\,{}_3F_2\!\left(1, \tfrac{1}{2}-\varepsilon, \tfrac{1}{2}-\varepsilon; \tfrac{3}{2}+\varepsilon, \tfrac{3}{2}; 1\right). \end{aligned} \tag{41}$$

Expanding the Γ-functions in the summand of the series, one obtains the ε-expansion of B_4 from boxed sums of the form

$$S_{\mathrm{odd}}(a; b_1, \ldots, b_{k-1}) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^a} \prod_{i=1}^{k-1} \sum_{m_i=1}^{n-1} \frac{1}{(2m_i-1)^{b_i}}, \tag{42}$$

which, like (24), is symmetric in $\{b_i\}$, though now it involves reciprocal powers of *odd* integers. To relate these to (1), one combines 2^k k-fold Euler sums with arguments of differing sign, to restrict the nested summations to odd integers. Symmetrizing over $(k-1)!$ permutations of all but the first argument, one exhausts those summations in (42) with distinct values of $\{m_i\}$. Adding combinations of Euler sums with lesser depth, one iteratively includes the degenerate summations. Hence the expansion of B_4 to $\mathrm{O}(\varepsilon^3)$ can be achieved in terms of the basis (17), by routine (and repeated) reference to the database.

The result is

$$
\begin{aligned}
B_4 \;=\;& \tfrac{1}{2}\left\{-13\,\zeta(4)+32\,\alpha(4)\right\}+\tfrac{1}{2}\varepsilon\left\{-239\,\zeta(5)+192\,\alpha(5)+204\ln 2\,\zeta(4)\right\}\\
&+\tfrac{3}{2}\varepsilon^2\left\{13\,\zeta(6)-74\,\zeta^2(3)+160\,U_{5,1}+384\,\alpha(6)-204\ln^2 2\,\zeta(4)\right\}\\
&+\tfrac{1}{72}\varepsilon^3\left\{-329385\,\zeta(7)+45853\,\zeta(3)\,\zeta(4)+10875\,\zeta(2)\,\zeta(5)+7680\,\zeta(5,1,-1)\right.\\
&-9280\,\zeta(3,3,-1)-11520\,\alpha(4)\,\zeta(3)+176031\ln 2\,\zeta(6)-2930\ln 2\,\zeta^2(3)\\
&\left.-48000\ln 2\,U_{5,1}+248832\,\alpha(7)+44064\ln^3 2\,\zeta(4)\right\}+\mathrm{O}(\varepsilon^4)\,. \qquad (43)
\end{aligned}
$$

Note that the polylogarithms (18) simplify the expansion; had one merely used the conventional [41] polylogarithms $\mathrm{Li}_n\left(\frac{1}{2}\right)$, there would have been 10 additional terms, to this order in the expansion. The terms involving $\ln^{l-4} 2\,\zeta(4)$ follow $\alpha(l)$, being generated by

$$
\sum_{l>4}(6\varepsilon)^{l-4}\left\{16\,\alpha(l)-\frac{17}{(l-4)!}(-\ln 2)^{l-4}\,\zeta(4)\right\}. \qquad (44)
$$

A strong check of (43) was performed, using the representation

$$
(1+2\varepsilon)\left[\frac{\Gamma(1-\varepsilon)}{\Gamma(1+\varepsilon)\,\Gamma(1-2\varepsilon)}\right]^2\int_0^1\int_0^1\frac{\mathrm{d}x\,\mathrm{d}y}{1-x^2y^2}\left[\frac{(1-x^2)(1-y^2)^2}{(8xy)^2}\right]^{\varepsilon} \qquad (45)
$$

of the hypergeometric series in (41), expanding the integrand to $\mathrm{O}(\varepsilon^5)$, and using the NAG routine D01FCF to evaluate 6 double integrals to 8 significant figures. By contrast, 800 significant figures were rapidly obtained by expressing sums of the form (42) in terms of those in the database. The agreement of NAG with the more powerful methods developed here gives one confidence in the computer-algebraic book-keeping that produced (43).

There is a further [42] non-trivial diagram entailed in three-loop charge renormalization: the two-loop fermion-propagator [43, 44] diagram, with a three-fermion intermediate state, contributing to on-shell mass renormalization [43, 45]. However, this is eventually expressible [38] in terms of Γ-functions and the same hypergeometric series as is encountered in (41). Thus no further analysis of irreducible Euler sums is entailed, though there is a great deal of book-keeping to perform, to obtain the three-loop terms in the dimensionally regularized charge-renormalization constant [38]

$$
Z_3=1+\sum_{n=1}^{\infty}C_n\left(\frac{e^2\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}m^{2\varepsilon}}\right)^n, \qquad (46)
$$

where e and m are the on-shell charge and mass.

The one-loop and two-loop contributions are easily found exactly [44]:

$$
C_1=-\frac{4}{3\varepsilon},\qquad C_2=-\frac{4(1+7\varepsilon-4\varepsilon^3)}{\varepsilon(2-\varepsilon)(1-2\varepsilon)(1+2\varepsilon)}\,. \qquad (47)
$$

The first three [38] terms in the Laurent expansion

$$
C_3=-\frac{8}{9\varepsilon^2}+\frac{62}{27\varepsilon}+\left\{128\ln 2\,\zeta(2)-\frac{22}{3}\,\zeta(3)-\frac{368}{3}\,\zeta(2)+\frac{4867}{81}\right\}+\sum_{n=1}^{\infty}C_{3,n}\varepsilon^n \qquad (48)
$$

are free of contributions from B_4, whose expansion starts at level $l=4$. Level-3 terms are sufficient for the analysis [46, 47, 48] of a restricted set of four-loop contributions to the anomalous magnetic moment of the muon [49], and for the study of quark-mass effects at the three-loop level of quantum chromodynamics [50]. Thanks to (43), the expansion (48) may now be continued to level 7, i.e. to the same level as is reached at four loops in the electron's anomaly, whose polylogarithms attain order $2L-1$ at L loops. After much computer algebra, I was able to obtain:

$$\begin{aligned}
C_{3,1} &= -384\ln^2 2\,\zeta(2)+\frac{1358}{3}\zeta(4)-\frac{736}{3}\alpha(4)+1024\ln 2\,\zeta(2)-\frac{2957}{6}\zeta(3)\\
&\quad -\frac{1960}{3}\zeta(2)+\frac{104113}{486} \qquad (49)\\
&\approx -179.724\,615\,842\,918\,120\,241\,823\,332\,320\,650\,562\,692\,071\,547\,121\,,\\
C_{3,2} &= 768\ln^3 2\,\zeta(2)-284\ln 2\,\zeta(4)+384\,\zeta(2)\,\zeta(3)+\frac{5101}{3}\zeta(5)-1472\,\alpha(5)\\
&\quad -3072\ln^2 2\,\zeta(2)+\frac{55571}{18}\zeta(4)-\frac{30488}{9}\alpha(4)+4096\ln 2\,\zeta(2)\\
&\quad -\frac{34537}{12}\zeta(3)-\frac{7324}{3}\zeta(2)+\frac{1937227}{2916} \qquad (50)\\
&\approx -427.138\,027\,736\,892\,466\,683\,630\,594\,488\,509\,635\,499\,554\,227\,264\,,\\
C_{3,3} &= -1152\ln^4 2\,\zeta(2)+852\ln^2 2\,\zeta(4)-1280\ln 2\,\zeta(2)\,\zeta(3)+\frac{5018}{3}\zeta^2(3)+6356\,\zeta(6)\\
&\quad -8832\,\alpha(6)-3680\,U_{5,1}+6144\ln^3 2\,\zeta(2)-\frac{34067}{3}\ln 2\,\zeta(4)+\frac{6272}{3}\zeta(2)\,\zeta(3)\\
&\quad +\frac{591473}{36}\zeta(5)-\frac{60976}{3}\alpha(5)-12288\ln^2 2\,\zeta(2)+\frac{1472549}{108}\zeta(4)\\
&\quad -\frac{450388}{27}\alpha(4)+13696\ln 2\,\zeta(2)-\frac{733013}{72}\zeta(3)-\frac{25226}{3}\zeta(2)+\frac{33051769}{17496} \qquad (51)\\
&\approx -1371.792\,496\,978\,355\,362\,371\,049\,120\,514\,541\,715\,942\,648\,466\,341\,,\\
C_{3,4} &= \frac{6912}{5}\ln^5 2\,\zeta(2)-1704\ln^3 2\,\zeta(4)+3840\ln^2 2\,\zeta(2)\,\zeta(3)+\frac{33695}{54}\ln 2\,\zeta^2(3)\\
&\quad -\frac{334273}{12}\ln 2\,\zeta(6)+\frac{92000}{9}\ln 2\,U_{5,1}+\frac{7360}{3}\alpha(4)\,\zeta(3)+\frac{68689}{36}\zeta(2)\,\zeta(5)\\
&\quad -\frac{334907}{108}\zeta(3)\,\zeta(4)+\frac{2459549}{36}\zeta(7)-52992\,\alpha(7)-\frac{14720}{9}\zeta(5,1,-1)\\
&\quad +\frac{53360}{27}\zeta(3,3,-1)-9216\ln^4 2\,\zeta(2)+34067\ln^2 2\,\zeta(4)-10240\ln 2\,\zeta(2)\,\zeta(3)\\
&\quad +\frac{129179}{6}\zeta^2(3)+33767\,\zeta(6)-121952\,\alpha(6)-\frac{152440}{3}U_{5,1}\\
&\quad +24576\ln^3 2\,\zeta(2)-\frac{1176869}{18}\ln 2\,\zeta(4)+\frac{21184}{3}\zeta(2)\,\zeta(3)+\frac{15720695}{216}\zeta(5)\\
&\quad -\frac{900776}{9}\alpha(5)-41088\ln^2 2\,\zeta(2)+\frac{27328097}{648}\zeta(4)-\frac{4616354}{81}\alpha(4)\\
&\quad +43712\ln 2\,\zeta(2)-\frac{14076461}{432}\zeta(3)-\frac{82735}{3}\zeta(2)+\frac{555842827}{104976} \qquad (52)\\
&\approx -3273.919\,335\,883\,520\,406\,469\,573\,320\,145\,714\,810\,021\,184\,454\,681\,.
\end{aligned}$$

Terms (49,50) are in agreement with the polylogarithms of order up to 5 that were obtained in [38], though they are much simpler in form, thanks to the use of (18).

Terms (51,52) are new, and show how complex perturbation expansions may become, when they entail polylogarithms of orders 6 and 7, as undoubtedly happens in the four-loop contributions to the electron's anomaly. However, I would be rather surprised were the four-loop anomaly to contain *further* transcendentals, not included above, since Euler sums are the natural structures to emerge from ε-expansions of generalized hypergeometric series [40, 51, 52] whose parameters differ from half-integers [38, 53, 54], or integers [8, 55, 56], by multiples of ε. In fact, there is good reason to suppose that not all of the terms above will occur in the four-loop anomaly, since Laporta and Remiddi [14] have shown that $\mathrm{Li}_5\left(\frac{1}{2}\right)$ is absent at three loops.

8 Consequences

The enumerations (9,10,16) have consequences for field theory, knot theory, and number theory, which will be discussed in that order.

8.1 Field theory

For calculational quantum field theorists, the enumeration (10) amounts to 'counting the enemy', since each new irreducible Euler sum corresponds to the existence of a class of polylogarithmic integrals that cannot be related to previously evaluated integrals, by computer-algebraic methods. They are few in number, at the levels where calculations are currently performed. Their tally, in Table 2, is the integer sequence M0317 of [18].

As reviewed in [57, 58, 59], there has been tremendous progress in the use of computer algebra, most notably by recursive methods within the framework of dimensional regularization, which automate the computation of single-scale massless [60, 61] and on-shell massive [14, 38, 48] propagator diagrams by symbolic manipulation of vast [62] numbers of polynomials in the analytically continued [39] dimensionality, $2\mu = 4 - 2\varepsilon$, of spacetime. The residual difficulty then resides in extracting the Laurent expansions, as $\varepsilon \to 0$, of a small set of irreducible Feynman integrals. Recent success at three-loops [14], with the electron's anomalous magnetic moment, provides an impressive example of the power of this technique. Following discussions at the AI-HEP-92 workshop [58], Laporta and Remiddi found it possible to achieve a computer-algebraic reduction of all 3-loop electron-anomaly diagrams to merely 18 terms [14], using integration by parts [60, 61] in $4 - 2\varepsilon$ dimensions. Drawing on experience of polylogarithmic integration, from previous 4-dimensional work, they were able to extract the requisite Laurent expansions as $\varepsilon \to 0$. Along with the inevitable depth-1 irreducibles, $\ln 2$ and $\{\,\zeta(n) \mid n \le 5\}$, their result contains just one irreducible double Euler sum, namely $U_{3,1}$, though it is presented in [14] in terms of the more conventional [41] polylogarithm $\mathrm{Li}_4\left(\frac{1}{2}\right)$, along with the precise $(\ln 2)^4$ and $\pi^2(\ln 2)^2$ terms of (6).

In the case of field-theory counterterms, up to L loops, the irreducible Euler sums

$$\{\,\zeta(2a+1) \mid L-2 \ge a > 0\}\,, \tag{53}$$

$$\{\,U_{2a+1,2b+1} \mid a > b > 0,\ L-3 \ge a+b\}\,, \tag{54}$$

$$\{\,\zeta(2a+1,2b+1,2c+1) \mid a \ge b \ge c > 0,\ a > c,\ L-3 \ge a+b+c\}\,, \tag{55}$$

appear. The first set, studied in [33, 34, 36], corresponds [3] to 2-braid torus knots, with the trefoil knot $3_1 \simeq \zeta(3)$ first appearing in counterterms at $L = 3$ loops. The second [16, 17] set corresponds [8] to a restricted set of positive 3-braids, with $8_{19} \simeq U_{5,3}$ first appearing at $L = 6$ loops, and the third [2, 5] to positive 4-braids, with the uniquely positive 11-crossing non-torus knot $\sigma_1^2\sigma_2^2\sigma_1\sigma_3\sigma_2^3\sigma_3^2 \simeq \zeta(5,3,3)$ first appearing at $L = 7$ loops. Hence the 5-loop renormalization of ϕ^4 theory was accomplished [63] in terms of only depth-1 irreducibles, whereas at 6 and 7 loops one first encounters [5] 2-fold and 3-fold Euler sums, respectively.

It is not yet known whether Euler sums exhaust the transcendentals in counterterms at $L \geq 7$ loops. In addition to $10_{124} = \sigma_2\sigma_1^5\sigma_2\sigma_1^3 \simeq U_{7,3}$, there are two further positive knots with 10 crossings, namely [32] $10_{139} = \sigma_2\sigma_1^3\sigma_2^3\sigma_1^3$ and $10_{152} = \sigma_2^2\sigma_1^2\sigma_2^3\sigma_1^3$, which 7-loop analysis [5] suggests are *not* associated with Euler sums of depth $k < 4$. All those, and *only* those, subdivergence-free 7-loop ϕ^4 diagrams whose link diagrams skein to 10_{139} and 10_{152} appear, on the basis of numerical evidence, to give counterterms that cannot be reduced to Euler sums with depth $k \leq 3$. Hence it is an open (and fascinating) question whether two[8] of the 5 new irreducibles in $S_{10,4}$ are associated with these knots.

Turning to the finite parts of Feynman diagrams, one learns from three-loop analyses [13, 14] that $U_{3,1}$ appears, via (37,38). Emphatically *no* claim is made that Euler sums exhaust the transcendentality content of perturbative quantum field theory; polylogarithms of non-trivial mass and momentum ratios are everywhere dense. In single-scale process, however, where such ratios are unity, or zero, it *may* occur that the results entail only Euler sums, as in the case of electron's anomaly, $\frac{1}{2}(g-2)_\mathrm{e}$ at three loops [14]. On the other hand, with a different configuration of unit and zero masses the maximum value of Clausen's integral [41] is often generated [42, 53].

The key to deciding whether a result is reducible to Euler sums is an analysis of the hypergeometric functions [8, 38, 53] ${}_{p+1}F_p(1, \{a_i\}; \{b_i\}; z)$ that are produced in $4 - 2\varepsilon$ dimensions. If $z = \pm 1$, and $\{a_i, b_i \mid i \leq p\}$ differ from integers, or half integers, by multiples of ε, then reducibility to Euler sums is guaranteed, as in (41,43). Such a reduction to hypergeometric series has been achieved for the electron anomaly, $\frac{1}{2}(g-2)_\mathrm{e}$, at two loops [38, 64], for the charge-renormalization constant, Z_3, at three loops [38], for the Gell-Mann–Low function of quantum electrodynamics to all orders in the large-N limit [11], and for a corresponding limit of the quantum chromodynamics of heavy-quark interactions [65]. That is the basis for my strong belief that a similar reduction to hypergeometric series, or some generalization [66, 67, 68, 69] of them, underlies the finding of Laporta and Remiddi that the three-loop anomaly involves just one Euler sum with depth $k > 1$, namely the very specific polylogarithmic combination (6). It is also reasonable to expect that this reducibility to Euler sums persists beyond three loops, where only numerical [70] estimates are currently available.

8.2 Knot theory

To plain knot theorists, the preoccupations of a knot/field-theorist [3, 4, 12] may appear rather restricted. The correspondence claimed by Dirk Kreimer is between *positive* knots and the numbers appearing in field-theory counterterms. His process of discovery began

[8] Recall that $U_{3,1}$ and $U_{5,1}$ are absent from counterterms; no positive knot has crossing number 4 or 6.

with the observation [3] that the removal of sub-divergences from Feynman diagrams, construed as a skein relation between link diagrams encoding the momentum flow, yields a counterterm that is rational if the skeining results in the unknot, as in the case of ladder diagrams, where cancellations of ζ-functions have been demonstrated perturbatively [4] and non-perturbatively [71]. It continued [4] with the observation that the 2-braid torus knot $(2L-3,2)$ is produced by skeining the crossed-ladder diagram that generates [33] $\zeta(2L-3)$ at $L \geq 3$ loops.

To continue the correspondence, it was clearly necessary to do two things: to go to loop-numbers, L, higher than the then current limit of $L = 5$ [63], which is how I became involved [5], and to enumerate positive knots, which we[9] have taught REDUCE [22] to do, up to 17 crossings, barring degeneracy of HOMFLY [32] polynomials. Here attention is largely restricted to knots with up to 13 crossings, as in Table 5. Knots with more crossings will figure in [12].

Table 5: Positive prime knots related to Euler sums, via field-theory counterterms.

crossings	knots	numbers
$2a+1$	σ_1^{2a+1}	$\zeta(2a+1)$
8	$\sigma_2\sigma_1^3\sigma_2\sigma_1^3 = 8_{19}$	$N_{5,3}$
9	none	none
10	$\sigma_2\sigma_1^5\sigma_2\sigma_1^3 = 10_{124}$	$N_{7,3}$
	$\sigma_2\sigma_1^3\sigma_2^3\sigma_1^3 = 10_{139}$	?
	$\sigma_2^2\sigma_1^2\sigma_2^3\sigma_1^3 = 10_{152}$	?
11	$\sigma_1^2\sigma_2^2\sigma_1\sigma_3\sigma_2^3\sigma_3^2$	$N_{3,5,3}$
12	$\sigma_2\sigma_1^7\sigma_2\sigma_1^3$	$N_{9,3}$
	$\sigma_2\sigma_1^5\sigma_2\sigma_1^5$	$N_{7,5} - \frac{\pi^{12}}{2^5 \cdot 10!}$
	$\sigma_2\sigma_1^3\sigma_2^5\sigma_1^3$	?
	$\sigma_2\sigma_1^3\sigma_2^3\sigma_1^5$	?
	$\sigma_2^2\sigma_1^2\sigma_2^3\sigma_1^5$	?
	$\sigma_2^2\sigma_1^3\sigma_2^3\sigma_1^4$	?
	$\sigma_2^3\sigma_1^3\sigma_2^3\sigma_1^3$	?
13	$\sigma_1\sigma_2^3\sigma_1^2\sigma_3\sigma_2^4\sigma_3^2$	$N_{3,7,3}$
	$\sigma_1^2\sigma_2\sigma_1^3\sigma_3^2\sigma_2^2\sigma_3^3$	$N_{5,3,5}$
	$\sigma_1\sigma_2^3\sigma_1\sigma_3\sigma_2^5\sigma_3^2$	?
	$\sigma_1^2\sigma_2\sigma_1^3\sigma_3^3\sigma_2\sigma_3^3$	?
	$\sigma_1\sigma_2^3\sigma_1\sigma_3^2\sigma_2^3\sigma_3^3$	?
	$\sigma_1^2\sigma_2^2\sigma_1(\sigma_3\sigma_2^3)^2$	?
	$(\sigma_2\sigma_1\sigma_3\sigma_2)^3\sigma_1$	?
	$(\sigma_2\sigma_1\sigma_3\sigma_2)^3\sigma_2$	?

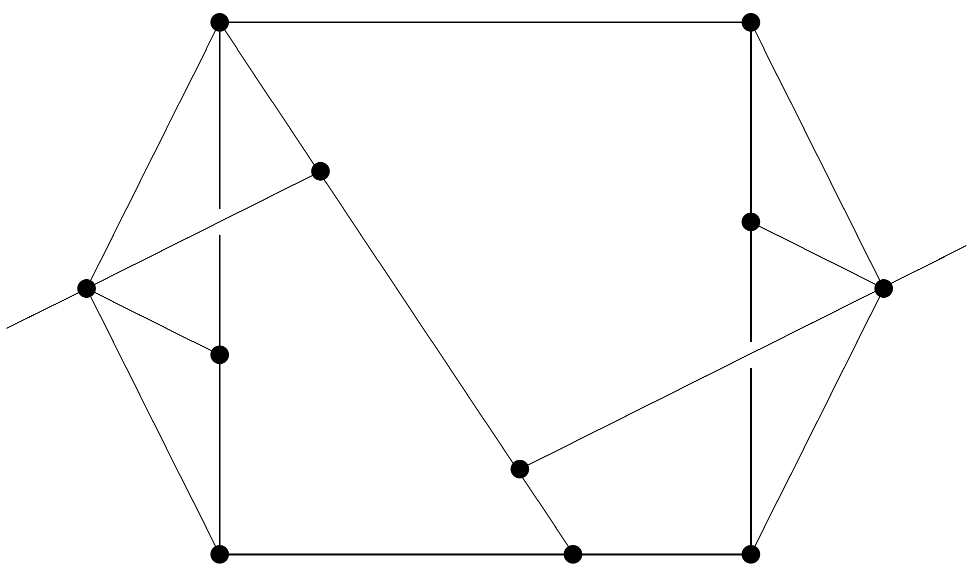
Excluding the 2-braid torus knots that correspond to depth-1 Euler sums, the tally of positive knots with 8 to 13 crossings is apparent in the knot column of Table 5. It forms the sequence 1, 0, 3, 1, 7, 8, ..., beginning at 8 crossings with the 3-braid torus knot 8_{19}. By contrast, the corresponding sequence for *all* prime knots with 8 to 13 crossings is given by M0851 of [18, 23] as 21, 49, 165, 552, 2176, 9988, ..., whose richness reveals the specificity of the preoccupations of knot/number/field theorists, to whom, of course, the

[9]Table 5 was obtained in collaboration with Dirk Kreimer, who will give further details in [12].

sparsity of positive knots is a delight: it is broadly commensurate with the slow growth in number of irreducible Euler sums, at corresponding levels.

There is no method, as yet, to assign numbers to knots, other than by brute-force evaluation of counterterms from diagrams whose skeinings produce the knots. The obstacle to high-precision evaluation of such counterterms, beyond 6 loops, is apparent from the fact that the knots 10_{139} and 10_{152} result, at 7 loops, from diagrams whose evaluation, via Gegenbauer-polynomial techniques [72], entails 7-fold summations weighted by the squares [73] of 6–j symbols, in such a manner as to make the evaluation-time of a truncation at N increase at least as fast as N^4. This is in stark contrast with the linear growth in Section 4.1, whose computational challenge hence pales into insignificance, compared with that required for [5].

Fig. 1: A diagram from cutting a 9-loop bubble that skeins to 13-crossing 4-braids.



Fortunately, there is a class of Feynman diagrams that may, with effort, be reduced to triple Euler sums, by analytical methods. A relatively simple example is provided by the 8-loop two-point diagram of Fig. 1, which is finite in 4 dimensions. With unit external momentum, a massless propagator $1/p_{\rm line}^2$ for each line, unit vertices, and euclidean measure $\pi^{-2}\int {\rm d}^4 p_{\rm loop}$ for each loop, this diagram was evaluated by REDUCE as

$$\begin{aligned} G(3,2,2) \; = \; & -405\,N_{3,7,3}+\tfrac{3675}{4}\,N_{2,9,2}+4680\,N_{2,7,2}+\tfrac{21285}{4}\,\zeta(13)-20535\,\zeta(11) \\ & +6480\,\zeta(9)\,\zeta(3)+12680\,\zeta(9)+480\,\zeta(7)\,\zeta(5)-19500\,\zeta(7)\,\zeta^2(3) \\ & -7200\,\zeta(7)\,\zeta(3)-79380\,\zeta(7)-14700\,\zeta^2(5)\,\zeta(3)-1200\,\zeta^2(5) \\ & -68160\,\zeta(5)\,\zeta^2(3)+38880\,\zeta(5)\,\zeta(3)-11520\,\zeta^3(3)+30240\,\zeta^2(3)\,, \quad (56) \end{aligned}$$

by means of the master formula [5, 15]

$$\begin{aligned} G(a,b,c) \; = \; & \sum_{i,j,k}\binom{2a-i}{a}\binom{2b-j}{b}\binom{2c-k}{c}\frac{(i+j+k)!}{i!j!k!}\sum_{p,m,n}\frac{\Delta(p,m,n)}{p^{2a-i}m^{2b-j}n^{2c-k}} \\ & \times\left[\left(\frac{2}{p+m+n-1}\right)^{i+j+k+1}+\left(\frac{2}{p+m+n+1}\right)^{i+j+k+1}\right], \qquad (57) \end{aligned}$$

where $\Delta(p,m,n)$ results from angular integrations over Chebyshev polynomials [72] and is 1, or 0, according as whether $g=(p+m+n+1)/2$ is, or is not, an integer satisfying $g>\max(p,m,n)$.

Each irreducible 3-fold sum appearing in (56) belongs to one of the two-parameter families:

$$\begin{aligned}
N_{2m+1,2n+1,2m+1} &= \zeta(2m+1,2n+1,2m+1)-\zeta(2m+1)\,\zeta(2m+1,2n+1)\\
&\quad+\sum_{k=1}^{m-1}\binom{2n+2k}{2k}\zeta_P(2n+2k+1,2m-2k+1,2m+1)\\
&\quad-\sum_{k=0}^{n-1}\binom{2m+2k}{2k}\zeta_P(2m+2k+1,2n-2k+1,2m+1)\,, \qquad (58)\\
N_{2m,2n+1,2m} &= \zeta(2m,2n+1,2m)+\zeta(2m)\left\{\zeta(2m,2n+1)+\zeta(2m+2n+1)\right\}\\
&\quad+\sum_{k=1}^{m-1}\binom{2n+2k}{2k}\zeta_P(2n+2k+1,2m-2k,2m)\\
&\quad+\sum_{k=0}^{n-1}\binom{2m+2k}{2k+1}\zeta_P(2m+2k+1,2n-2k,2m)\,, \qquad (59)
\end{aligned}$$

with product terms

$$\zeta_P(a,b,c)=\zeta(a)\left\{2\,\zeta(b,c)+\zeta(b+c)\right\}\,, \qquad (60)$$

whose systematic inclusion, with the combinatoric factors in (58,59), removes all trace of the non-knot number π^2 from (56), and likewise from *every* diagram $G(a,b,c)$ with $a+b+c\leq 11$, according to similar, but *much* lengthier, evaluations of two-point functions obtained by cutting bubble diagrams with up to 13 loops, corresponding to knots with up to 23 crossings.

In general, subdivergence-free bubble diagrams, with up to L loops, yield the knot-numbers $\{\,N_{a+2,2b+1,a+2}\mid a\geq 0,\ b\geq 0,\ L-4\geq a+b\}$ as the very specific combinations (58,59) of Euler sums with depths $k\leq 3$ and levels $l\leq 2L-3$ [5]. They are knot-numbers, in the sense of [3, 4, 5], because counterterms, from subdivergence-free diagrams that skein to produce the corresponding knots, contain these numbers, and products of other knot-numbers, corresponding to factor [30] knots. The counterterms do *not* contain the non-knot irreducibles $\ln 2$ and π^2. Thus the combinations of Euler sums in (58,59) provide log-free and π-free bases for search spaces in which to evaluate classes of counterterms from diagrams that skein to 4-braids. For example, it was possible [5] to evaluate all 7-loop ϕ^4 counterterms from subdivergence-free diagrams that skein to 4-braids in terms of just 3 knot-numbers: $\zeta(11)$, $\zeta(5)\,\zeta^2(3)$, and $N_{3,5,3}+7\,\zeta(5)\,\zeta^2(3)=\zeta(3,5,3)-\zeta(3)\,\zeta(5,3)$, corresponding to the 2-braid torus knot $(11,2)$, the factor knot $5_1\times 3_1\times 3_1$, and the uniquely positive 11-crossing hyperbolic[10] knot $\sigma_1^2\sigma_2^2\sigma_1\sigma_3\sigma_2^3\sigma_3^2$. The factor knot $3_1\times 8_{19}$ is not produced by skeining the link diagrams that encode the momentum flow. However, every diagram that skeins to $\sigma_1^2\sigma_2^2\sigma_1\sigma_3\sigma_2^3\sigma_3^2$ also produces the other two knots, $(11,2)$ and $5_1\times 3_1\times 3_1$. Until a method is devised to predict the rational coefficients with which knot-numbers occur, rather than determining them empirically, as at present, the association $\sigma_1^2\sigma_2^2\sigma_1\sigma_3\sigma_2^3\sigma_3^2\simeq N_{3,5,3}$ can be made only modulo $\zeta(11)$ and $\zeta(5)\,\zeta^2(3)$ terms. On this understanding, $N_{3,5,3}$ appears in the 11-crossing entry of Table 5.

[10]In [5], this knot, 10_{139}, and 10_{152}, were wrongly called satellite knots; all three are, in fact, hyperbolic.

In Table 5, $N_{3,5,3}$ is the *first* irreducible triple sum to appear, at 11 crossings. That is because the knot-numbers (58,59) are not all independently irreducible. Up to level 13, the following odd-zeta reductions are obtained, by extending the methods of [2] to (57):

$$\begin{aligned}
N_{2,1,2} &= \tfrac{9}{2}\,\zeta(5)\\
N_{2,3,2} &= \tfrac{75}{8}\,\zeta(7)\\
N_{3,1,3} &= -\tfrac{1}{4}\,\zeta(7)\\
N_{2,5,2} &= \tfrac{439}{36}\,\zeta(9)+\tfrac{8}{3}\,\zeta^3(3)\\
N_{3,3,3} &= \tfrac{1}{3}\,\zeta(9)-\tfrac{4}{3}\,\zeta^3(3)\\
N_{4,1,4} &= \tfrac{115}{18}\,\zeta(9)-\tfrac{4}{3}\,\zeta^3(3)\\
4\,N_{3,5,3}-5\,N_{2,7,2} &= -\tfrac{1031}{24}\,\zeta(11)-88\,\zeta(5)\,\zeta^2(3)\\
N_{4,3,4}-5\,N_{2,7,2} &= \tfrac{103}{8}\,\zeta(11)-80\,\zeta(5)\,\zeta^2(3)\\
2\,N_{5,1,5}-5\,N_{2,7,2} &= -\tfrac{1091}{24}\,\zeta(11)-56\,\zeta(5)\,\zeta^2(3)\\
32\,N_{4,5,4}+140\,N_{3,7,3}-525\,N_{2,9,2} &= \tfrac{24425}{3}\,\zeta(13)-12880\,\zeta(7)\,\zeta^2(3)-8400\,\zeta^2(5)\,\zeta(3)\\
64\,N_{5,3,5}-100\,N_{3,7,3}+175\,N_{2,9,2} &= -673\,\zeta(13)+4400\,\zeta(7)\,\zeta^2(3)+3440\,\zeta^2(5)\,\zeta(3)\\
8\,N_{6,1,6}-12\,N_{3,7,3}+49\,N_{2,9,2} &= -115\,\zeta(13)+976\,\zeta(7)\,\zeta^2(3)+752\,\zeta^2(5)\,\zeta(3)\,.
\end{aligned}\tag{61}$$

At level $l=2L-3>3$, corresponding to $L>3$ loops, there are $L-3$ knot-numbers of the form $\{\,N_{a+1,2b+1,a+1}\mid a>0,\ b\geq 0,\ a+b=L-3\}$, while (55) gives $\lceil (L-3)^2/12\rceil-1$ irreducibles [2]. Hence the knot-numbers (58,59) fail to exhaust the irreducibles, for $L\geq 16$ loops. Up to $L=13$ loops, their sufficiency has been proven, using REDUCE.

Two independent irreducibles, chosen to be $N_{3,7,3}$ and $N_{5,3,5}$, appear in the 13-crossing part of Table 5. They are associated with the first two braid words, on the basis of intensive skeining of the link diagrams that encode momentum flows in bubble diagrams. A hint of the pen-and-paper labour undertaken by Dirk Kreimer is given by his drawings in [8], which refer to much simpler Feynman diagrams. Since at least two 13-crossing knots of Table 5 emerge from skeining any diagram that yields irreducible triple sums of level 13, we cannot yet determine which rational combination of the two irreducibles is associated with a given knot. By a combination of skeining and inspection of factorizations of Alexander [32] polynomials, we arrive at the association of the first two 13-crossing knots of Table 5 with, as yet undetermined, linear combinations of $N_{3,7,3}$ and $N_{5,3,5}$. The remaining 6 knot-numbers at level 13 cannot all come from $S_{13,3}$, since Table 1 reveals that $E(13,3)=7$, of which the two non-alternating irreducibles have already been accounted for. It is possible that further 13-crossing knot-numbers come from $S_{13,5}$, whose size, $S(13,5)=133$, puts it out of the reach of MPPSLQ for the foreseeable future.

At even levels, the associations of Table 5 are made with combinations

$$N_{a,b}=\zeta(-a,b)-\zeta(-b,a)\,,\tag{62}$$

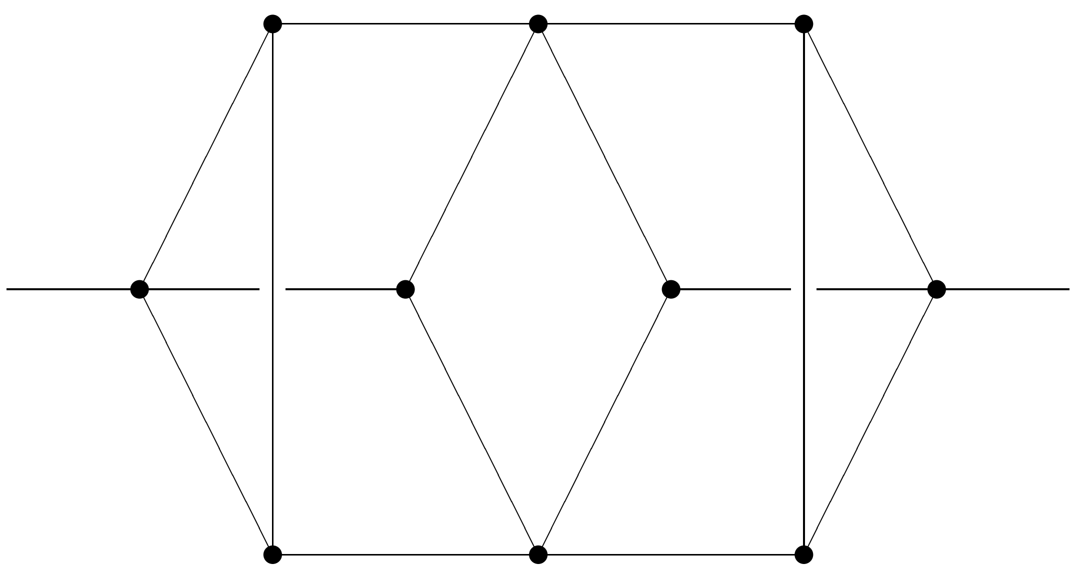
of *alternating* double Euler sums. The very simple constructs $N_{5,3}$ and $N_{7,3}$ remove $\{\pi^{2n}\mid n\leq 5\}$ from diagrams with up to 7 loops that skein to 8_{19} and 10_{124}. Modulo terms corresponding to factor knots, these two knot-numbers agree with the findings of [5], where results were written, equivalently, in terms of $29\,\zeta(8)-12\,\zeta(5,3)$ and $94\,\zeta(7,3)-793\,\zeta(10)$. Previously, the numbers in these combinations appeared gratuitous; now they are seen as consequences of ignoring the wider world of alternating sums.

Calculations at higher loops reveal that $N_{2a+5,3}$ is a knot-number at level $2a+8$. However, a multiple of $\pi^{2(a+b+6)}$ must be subtracted from $N_{2a+7,2b+5}$ to obtain a knot-number for 3-braids with 12 or more crossings. Table 5 shows that this subtraction has a very simple form at 12 crossings. The discovery of the general form of the subtraction is frustrated by the fact that counterterms associated with even-crossing knots are generally much more difficult to calculate than those associated with odd numbers of crossings, as witnessed by the table of results in [5], where 10_{139} and 10_{152} conspired to frustrate the precise evaluation of all subdivergence-free contributions to the 7-loop β-function of ϕ^4-theory. As these knots entail computation times scaling as N^4 for truncation at N, they leave us with a 7-loop result that is known to 'only' 11 significant figures, after 10^3 CPUhours. By way of a hard-won, but relatively simple, exact numerical result at 8 loops,

$$M(2,2,1,1)=2^{14}\cdot 3\,N_{9,3}+2^4\cdot 3\cdot 5^3\cdot 7\,\zeta(3)\,\zeta(9)-2^5\cdot 3^3\,\zeta^4(3)-2^3\cdot 3^2\cdot 577\,\zeta(5)\,\zeta(7) \quad (63)$$

is offered, as the finding of MPPSLQ for the Feynman diagram of Fig. 2.

Fig. 2: A diagram from cutting an 8-loop bubble that skeins to 12-crossing 3-braids.



The evaluation (63) was accomplished by using 4-dimensional Chebyshev-polynomial expansions, derived in [36], for the iteratively-defined coordinate-space constructs

$$P_{n+1}(x,y)=\int\frac{\mathrm{d}^4z}{\pi^2z^2}\,P_n(x,z)\,P_0(z,y)\,;\quad P_0(x,y)=1/(x-y)^2\,, \quad (64)$$

which are then combined by REDUCE to perform a radial integration in

$$M(a_1,a_2,a_3,a_4)=x^4\int\frac{\mathrm{d}^4y}{\pi^2}\prod_i P_{a_i}(x,y)=\sum_{n_i}A\left(\{n_i\}\right)\,R\left(\{a_i\};\{n_i\}\right)\,, \quad (65)$$

giving [15] a 4-fold sum, whose radial term, R, is a huge expression involving inverse powers of n_i and $h=\frac{1}{2}\sum_i n_i$, while the angular term, A, vanishes unless h is an integer greater than any n_i, in which case A is the smallest of the 8 values n_i and $h-n_i$. Taking

the REDUCE result, TRANSMP produces MPFUN code that implements the brute-force accelerator (25), with demon-number $d = 2$, yielding enough significant figures for MPPSLQ to discover (63). It would be very hard to obtain such results analytically. As so often, in field theory, the whole is much simpler than the parts: thousands of 4-fold non-Euler sums produce a one-line, π-free result, in terms of the alternating double-sum knot-number $N_{9,3}$, which cannot be expressed in terms of non-alternating sums. This led me to seek and find such things as (26), which in turn led to the enumeration (9).

Clearly there is a pressing need for a more developed knot/number/field theory, which might tell one which Euler (or other) sums in counterterms to associate with which knots, without need of laborious calculations of Feynman diagrams. In particular I would dearly like to know whether the two undetermined knot-numbers at level 10 are irreducible alternating 4-fold Euler sums, residing in $S_{10,4}$. Analytical assistance is ardently sought; without it, additional numerical work is likely to add little understanding.

Eventually, it may prove possible to relate Euler sums to positive knots, directly, in a way that is consistent with the field-theory route, yet does not oblige one to follow it. That is, I suggest, a substantial task, since it is notoriously difficult to derive non-trivial statements that apply to all members of a well-defined class of knot. For example, the enumeration, by REDUCE, of positive knots up to 17 crossings, is insecure against the possibility that two distinct positive knots might have the same HOMFLY polynomial, though no example of positive mutation has come to light.

It may be that the calculational complexities of field-theory counterterms, and the classificational complexities of knot theory, are mutual echoes, with which the now-apparent combinatoric simplicity of the enumeration of irreducible Euler sums eventually fails to resonate. For the present, however, the following correspondences, from computations to 13 loops, are rather impressive, to my mind.

1. Barring cancellations between diagrams, associated with dynamical symmetries, such as local gauge invariance [6] or supersymmetry [8], the Euler sum $\zeta(2L-3)$, corresponding [4] to the 2-braid torus knot $(2L-3, 2)$, first appears in anomalous dimensions at L loops. No other irreducible results from subdivergence-free diagrams with less than 6 loops, because no other positive knot has less than 8 crossings.

2. At 6 loops, 3-braids start to appear. The first of these is $8_{19} \simeq N_{5,3}$. The irreducibility of its knot-number was confirmed in [1], with no knowledge of prior developments in field theory [16, 17].

3. At 7 loops, 4-braids start to appear. The first of these is $\sigma_1^2\sigma_2^2\sigma_1\sigma_3\sigma_2^3\sigma_3^2 \simeq N_{3,5,3}$. The irreducibility of its knot-number was confirmed in [2], following communication of its appearance in field theory [5]. The 3-braid $10_{124} \simeq N_{7,3}$ is also encountered at 7 loops [5], in accord with the tally of [1].

4. At 8 loops, there appear: a pair of 13-crossing 4-braids, with knot-numbers $N_{3,7,3}$ and $N_{5,3,5}$, in *accord* with the tally of [2]; and a pair of 12-crossing 3-braids, with knot-numbers $N_{9,3}$ and $N_{7,5}$ modulo π^{12}, in *excess* of the tally of *non*-alternating double sums in [1]. The latter pair led, via (63), to (26), which resolved the apparent conflict between knot/field theory and number theory and then produced the enumerations (9,10), as contributions from mathematical physics to pure mathematics.

5. Results up to 13 loops confirm the association of the knot-numbers (58,59) with 4-braids, up to 23 crossing.

6. Results on 14-crossing knots, appearing at 9 loops, will be given in detail in [12]. They confirm the appearance of the expected pair of double-sum irreducibles, $N_{11,3}$ and $N_{9,5}$ modulo π^{14}. For the first time, a *truly* irreducible 4-fold Euler sum is obtained from a Feynman diagram. The associated knot-number is $\zeta(5,3,3,3)+\zeta(3,5,3,3)-\zeta(3)\,\zeta(5,3,3)$ modulo π^{14}.

8.3 Number theory

Mathematicians have only recently, it appears, made significant extensions of Euler's original study of double sums [74]. Alternating double sums, familiar in field theory [37] since [16], were studied in [1]; triple sums, encountered in [17], were studied in [2, 75]; generic non-alternating sums were studied in [27], where the sum rule (23) was obtained.

The extension of these studies into the entire domain of k-fold Euler sums, at all levels l, with all possible alternations of sign, was undertaken, in this work, in an unashamedly experimental[11] manner, stemming from an urgent need further to develop the connection between knot theory and quantum field theory.

Mathematics, at its purest, relies on little more than the fertile invention of the human mind. Mathematical physics often spawns structures of even greater beauty, thanks to what Einstein called the 'incomprehensible' comprehensibility of the natural world. As a new variation on this oft-repeated theme, field theory has led, from the observation [3] of the rationality of ladder-diagram counterterms, and the skeining of zeta-rich crossed-ladder diagrams [4], to a uniquely positive hyperbolic 11-crossing knot in 7-loop [5] counterterms, and in this work to the Feynman diagram of Fig. 2, whose evaluation (63) then resulted in the purely mathematical discovery that non-alternating Euler sums require alternating sums for their reduction, as witnessed by (26). Setting field theory aside, for a brief while, the larger universe of alternating Euler sums proved much easier to enumerate than its non-alternating restriction, as witnessed by (9).

The validity of the enumeration (9) is provisional: a year of work and 10^3 CPUhours, of the most exhaustive tests of which I and the engines at my disposal are capable, fail to reveal the slightest flaw in it. I invite colleagues with larger numerical appetites to test it further, in the lively expectation that it will survive.

What is needed now is the closest thing to proof for which it is reasonable to hope: the establishment by deductive methods of the validity of (9) as an *upper bound* on the number of irreducible k-fold Euler sums at level l. The more ambitious aim of proving it as an identity is unrealistic, until someone develops the machinery for proving, *inter alia*, the irrationality of $\zeta^2(137)/\zeta(274)$, and a denumerable infinity of suchlike things. The more modest proposal of proving that the number of irreducibles is *no more* than that given by (9) seems eminently realistic. To anyone disposed to undertake it, I offer the following informal restatement:

[11] See [76] for a suggestion of a working definition of *experimental* mathematics.

1. For concrete values of l and k, such that $l+k$ is even, and $l \geq k \geq 1$, form all the partitions of l into precisely k odd integers.

2. For each partition p_i, count the distinguishable permutations of these odd integers and denote the answer by P_i.

3. Let A_i be the number of products of lower-level irreducibles, associated with partition p_i by the generator (12) operating on already established irreducibles.

4. Summing $E_i = P_i - A_i$ over the odd partitions one arrives at (9).

By way of example, consider $S_{10,4}$, which admits of the odd partitions $p_1 = 7+1+1+1$, $p_2 = 3+3+3+1$, and $p_3 = 5+3+1+1$, with $P_1 = P_2 = 4!/3! = 4$ and $P_3 = 4!/2! = 12$. To exemplify the structure, only the argument strings of Section 6 are notated, with $(\overline{1})$ standing for $-\ln 2$. Thus the 3 products

$$(7)(\overline{1})(\overline{1})(\overline{1}),\ (\overline{7},\overline{1})(\overline{1})(\overline{1}),\ (7,\overline{1},\overline{1})(\overline{1}), \tag{66}$$

leave $E_1 = 4 - 3 = 1$ as the number of irreducibles associated with p_1. Similarly,

$$(3)(3)(3)(\overline{1}),\ (\overline{3},\overline{1})(3)(3),\ (3,\overline{1},\overline{3})(3), \tag{67}$$

leave $E_2 = 4 - 3 = 1$ irreducibles. Finally, the 9 products

$$\begin{aligned}&(5)(3)(\overline{1})(\overline{1}),\ (\overline{5},\overline{3})(\overline{1})(\overline{1}),\ (\overline{5},\overline{1})(3)(\overline{1}),\ (\overline{3},\overline{1})(5)(\overline{1}),\ (\overline{3},\overline{1})(\overline{5},\overline{1}),\\ &(5,\overline{1},\overline{1})(3),\ (3,\overline{1},\overline{1})(5),\ (5,\overline{1},\overline{3})(\overline{1}),\ (3,\overline{1},\overline{5})(\overline{1}),\end{aligned} \tag{68}$$

leave $E_3 = 12 - 9 = 3$ irreducibles. The tally $E(10,4) = 1+1+3 = 5$ agrees with (9). Note that in (66,67,68) the specific choices of Section 6 were made, for irreducibles in spaces of lesser depth and level. However, that was not necessary: all one needs is the *number* of irreducibles, associated with the sub-partition in the smaller space, which is itself determined by the Aufbau. By the same token, all one needs to carry forward from $S_{10,4}$, for later stages of the Aufbau, are the numbers, E_i, of irreducibles associated with each of the 3 partitions of 10 into 4 odd integers; it is not necessary to specify a concrete basis by choosing signs or specific orderings of arguments. Note also that the total number of permutations is merely an element of Pascal's triangle: $\sum_i P_i = \binom{\frac{l+k-2}{2}}{k-1}$, in general, with $4+4+12 = \binom{6}{3}$ in $S_{10,4}$. It is, therefore, an elementary exercise to build up Euler's triangle of Table 1, by repeated application of $E_i = P_i - A_i$, which is, so to speak, the microscopic version of the macroscopic results (9,10,16). To paraphrase:

> The number of irreducibles is given, by *Euler's* triangle, as the deficit between the number that *Pascal's* triangle generates, by permutation of integers in odd partitions, and the number of products *already* given by previous deficits.

In its final form, a proof of (9), as a rigorous upper bound, may appear almost as simple as the "Euler = Pascal – Already" paraphrase of what is to be proved. It may, however, require formidable combinatoric intuition to generate and, more importantly, to *organize* sufficient permutation and partial-fraction relations, between k-fold Euler sums, with all possible alternations of sign, as effectively as the authors of [1, 2] organized the relations between *non*-alternating sums at depths $k = 2$ and $k = 3$. Mindful of what was entailed by these lesser tasks, I did not attempt the greater.

8.4 Conclusion

I marvel that the quantum electrodynamics [77] of Dyson, Feynman, Schwinger, and Tomonaga [78] leads, after 50 years of dedicated calculation and the attainment of equally impressive experimental accuracy [79], to agreement [14] between theory and experiment on the *eleventh* significant figure in the magnetic moment of the electron. That skillful calculator of double sums, Leonhard Euler [80], would smile, one feels, on seeing nothing more complicated than $\sum_{n>m}(-1)^{n+m}/n^3m$ in the sixth-order perturbation expansion. Gauss, too, might be amused to see numbers, from his hypergeometric series [38, 40], and knots, whose codification [30] he began, walk hand in hand, down the perturbation expansions of quantum field theory, to all [8] orders in the coupling constant.

Such reassuring order, in the mathematical description of nature, at astonishing levels of accuracy, reinforced by experience [8, 11, 16, 17, 38, 47, 53, 54] with hypergeometric series generated by dimensional regularization, and now by the remarkable matches between Euler sums [1, 2] and Dirk Kreimer's knot/number/field theory [3, 4, 5, 6, 8], leads me to believe that the $k \leq 3$ sums in (17), and hence in (52), more than suffice for the electron's anomaly at eighth order. Certainly, they suffice for the reduction of all Euler sums with levels $l \leq 7$. At any level, one has only to consult (9) for the tally of irreducible k-fold Euler sums, be they construed as calculational obstacles or mathematical friends.

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References

[1] D. Borwein, J. M. Borwein and R. Girgensohn, *Proc. Edin. Math. Soc.* **38** (1995) 273.

[2] J. M. Borwein and R. Girgensohn, CECM research report 95-053 (1995), submitted to *Electronic J. Combinatorics*, with an appendix by D. J. Broadhurst.

[3] D. Kreimer, University of Tasmania preprint UTAS-PHYS-94-25 (1994), to appear in *J. Knot Theory and its Ramifications*.

[4] D. Kreimer, *Phys. Lett.* **B354** (1995) 117.

[5] D. J. Broadhurst and D. Kreimer, *Int. J. Mod. Phys.* **C6** (1995) 519.

[6] D. J. Broadhurst, R. Delbourgo and D. Kreimer, *Phys. Lett.* **B366** (1996) 421.

[7] D. J. Broadhurst and A. L. Kataev, *Phys. Lett.* **B315** (1993) 179.

[8] D. J. Broadhurst, J. A. Gracey and D. Kreimer, Open University preprint OUT–4102–46 (1996), in preparation.

[9] A. N. Vasil'ev, Yu. M. Pis'mak and J. R. Honkonen, *Theor. Math. Phys.* **46** (1981) 157; **47** (1981) 465; **50** (1982) 127.

[10] J. A. Gracey, *Mod. Phys. Lett.* **A7** (1992) 1945.

[11] D. J. Broadhurst, *Z. Phys.* **C58** (1993) 339.

[12] D. Kreimer, *Knots and Feynman Diagrams* (Cambridge University Press, in preparation).

[13] L. Avdeev, J. Fleischer, S. Mikhailov and O. Tarasov, *Phys. Lett.* **B336** (1994) 560.

[14] S. Laporta and E. Remiddi, Bologna preprint DFUB 96-05 (1996); hep-ph/9602417.

[15] D. J. Broadhurst, Open University report OUT-4102-18 (1985).

[16] D. J. Broadhurst, *Z. Phys.* **C32** (1986) 249.

[17] D. T. Barfoot and D. J. Broadhurst, *Z. Phys.* **C41** (1988) 81.

[18] N. J. A. Sloane and S. Plouffe, *The Encyclopedia of Integer Sequences* (Academic Press, San Diego, 1995).

[19] B.-S. Du, *Fibonacci Quart.* 27 (1989) 116.

[20] H. R. P. Ferguson and D. H. Bailey, NASA Ames Technical Report RNR-91-032 (1991).

[21] D. H. Bailey, *ACM Trans. Math. Software* **21** (1995) 379.

[22] A. C. Hearn, REDUCE User's Manual Version 3.5, Rand Publication CP78 (1993).

[23] N. J. A. Sloane, *Electronic J. Combinatorics* **1** (1994) F1.

[24] M. A. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965) 23.1.30.

[25] D. H. Bailey, *ACM Trans. Math. Software* **19** (1993) 288.

[26] D. H. Bailey, NASA Ames Technical Report RNR-90-022 (1990).

[27] A. Granville, University of Georgia report, unpublished.

[28] D. H. Bailey, J. M. Borwein and R. Girgensohn, *Experimental Math.* **3** (1994) 17.

[29] W. H. Press, *Numerical Recipes FORTRAN Book* (Cambridge University Press, 1992).

[30] C. C. Adams, *The Knot Book* (W. H. Freeman, New York, 1994).

[31] D. Rolfsen, *Knots and Links* (Publish or Perish, Berkeley, 1976).

[32] V. F. R. Jones, *Annals of Math.* **126** (1987) 335.

[33] V. V. Belokurov and N. I. Ussyukina, *J. Phys.* **A16** (1983) 2811.

[34] D. J. Broadhurst, *Phys. Lett.* **B164** (1985) 356.

[35] N. I. Ussyukina and A. I. Davydychev, *Phys. Lett.* **B305** (1993) 136; **B298** (1993) 363.

[36] D. J. Broadhurst, *Phys. Lett.* **B307** (1993) 132.

[37] V. A. Smirnov, *Renormalization and Asymptotic Expansions* (Birkhäuser, Basel, 1991).

[38] D. J. Broadhurst, *Z. Phys.* **C54** (1993) 599.

[39] G. 't Hooft and M. Veltman, *Nucl. Phys.* **B44** (1972) 189; **B50**(1972) 318.

[40] G. H. Hardy, *Proc. Camb. Phil. Soc.* **21** (1923) 492.

[41] L. Lewin, *Polylogarithms and Associated Functions* (North Holland, New York, 1981).

[42] D. J. Broadhurst, *Z. Phys.* **C47** (1990) 115.

[43] N. Gray, D. J. Broadhurst, W. Grafe and K. Schilcher, *Z. Phys.* **C48** (1991) 673.

[44] D. J. Broadhurst, N. Gray and K. Schilcher, *Z. Phys.* **C52** (1991) 111.

[45] D. J. Broadhurst and A. G. Grozin. *Phys. Rev.* **D52** (1995) 4082.

[46] D. J. Broadhurst, A. L. Kataev and O. V. Tarasov, *Phys. Lett.* **B298** (1993) 445.

[47] P. A. Baikov and D. J. Broadhurst, *New Computing Techniques in Physics Research IV*, ed. B. Denby and D. Perret-Gallix (World Scientific, Singapore, 1996) p 167.

[48] P. A. Baikov, Moscow State University preprint INP-96-10/417 (1996); hep-ph/9603267.

[49] T. Kinoshita, *Phys. Rev.* **D47** (1993) 5013.

[50] K. G. Chetyrkin, J. H. Kühn and M. Steinhauser, Karlsruhe preprint TTP95-41 (1995); hep-ph/9511430.

[51] W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge University Press, 1935).

[52] L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, 1966).

[53] D. J. Broadhurst, J. Fleischer and O. V. Tarasov, *Z. Phys.* **C60** (1993) 287.

[54] D. J. Broadhurst, P. A. Baikov, V. A. Ilyin, J. Fleischer, O. V. Tarasov and V. A. Smirnov, *Phys. Lett.* **B329** (1994) 103.

[55] S. J. Hathrell, *Ann. Phys.* **139** (1982) 136; **142** (1982) 34.

[56] A. V. Kotikov, Annecy preprint ENSLAPP-A-568/95 (1995); hep-ph/9512270.

[57] D. J. Broadhurst, in *Proceedings of the Seventh International Seminar, Quarks-92*, ed. D. Yu. Grigoriev, V. A. Matveev, V. A. Rubakov and P. G. Tinyakov (World Scientific, Singapore, 1993) pp 225-241.

[58] D. J. Broadhurst, in *New Computing Techniques in Physics Research II*, ed. D. Perret-Gallix (World Scientific, Singapore, 1992) pp 579-584.

[59] D. J. Broadhurst, in *New Computing Techniques in Physics Research III*, ed. K.-H. Becks and D. Perret-Gallix (World Scientific, Singapore, 1994) pp 511-516.

[60] K. G. Chetyrkin and F. V. Tkachov, *Nucl. Phys.* **B192** (1981) 159.

[61] F. V. Tkachov, *Phys. Lett.* **B100** (1981) 65.

[62] S. A. Larin, T. van Ritbergen and J. A. M. Vermaseren, *Nucl. Phys.* **B438** (1995) 278.

[63] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K. G. Chetyrkin and S. A. Larin, *Phys. Lett.* **B272** (1991) 39.

[64] J. Fleischer and O. V. Tarasov, *Phys. Lett.* **B283** (1992) 129; *Comput. Phys. Commun.* **71** (1992) 193.

[65] M. Beneke and V. M. Braun, *Nucl. Phys.* **B426** (1994) 301.

[66] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques* (Gauthier-Villars, Paris, 1926).

[67] H. Exton, *Multiple Hypergeometric Functions and Applications* (Ellis Horwood, Chicester, 1976).

[68] H. M. Srivastava and P. W. Karlson, *Multiple Gaussian Hypergeometric Series* (Wiley, New York, 1985).

[69] S. Bauberger, F. A. Berends, M. Böhm and M. Buza, *Nucl. Phys.* **B434** (1995) 383.

[70] T. Kinoshita, *IEEE Trans. Instrum. Meas.* **44** (1995) 498.

[71] R. Delbourgo, A. Kalloniatis and G. P. Thompson, University of Tasmania preprint, in preparation.

[72] K. G. Chetyrkin, A. L. Kataev and F. V. Tkachov, *Nucl. Phys.* **B174** (1980) 345.

[73] B. G. Nickel, *J. Math. Phys.* **19** (1978) 542.

[74] L. Euler, *Opera Omnia*, Ser. 1, Vol. XV (Teubner, Berlin, 1917) pp 217-267.

[75] C. Markett, *J. Number Theory* **48** (1994) 113.

[76] J. M. Borwein, P. B. Borwein, R. Girgensohn and S. Parnes, CECM research report 95-032, to appear in *Mathematical Intelligencer*.

[77] T. Kinoshita, *Quantum Electrodynamics* (World Scientific, Singapore, 1990).

[78] S. S. Schweber, *QED and the men who made it* (Princeton University Press, 1994).

[79] R. S. Van Dick, P. B. Schwinberg and H. G. Dehmet, *Phys. Rev. Lett.* **59** (1987) 26.

[80] L. Euler, *Novi Comm. Acad. Sci. Petropol.* **20** (1775) 140.